Coordinated replenishment policies

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M.J.G. van Eijs

Stellingen

behorende bij het proefschrift

Coordinated Replenishment Policies

van

M.J.G. van Eijs

8 oktober 1993

1. Logistiek is maatwerk. De in dit proefschrift bestudeerde bestelstrategieën en de gehanteerde oplossingsmethoden zijn derhalve slechts bruikbaar in praktische situaties nadat bedrijfs-afhankelijke specificaties zijn aangebracht.

2. De oplossingsmethode van Goyal (1974), die in de literatuur wordt gebruikt voor de bepaling van de optimale parameters van de indirekte groeperingsstrategie, leidt niet altijd tot het minimum van doelfunctie (2.1). (Zie hoofdstuk 3).

   - GOYAL, S.K., Determination of optimum packaging frequency of items jointly replenished, *Management Science* 21, 1974, 436-443.

3. Relatieve waarden, behorende bij een gegeven bestelstrategie, zijn zeer nuttig om inzicht te verkrijgen in de toekomstige kostenverschillen die ontstaan door op een besteltijdstip af te wijken van de basis bestelstrategie om op deze manier gebruik te maken van eventuele kostenvoordelen waarmee deze bestelstrategie geen rekening houdt. (Zie hoofdstuk 4 en 7).

4. De decompositie-benadering, die in de literatuur wordt gebruikt voor de bepaling van de optimale parameters van de can-order strategie, leidt tot een zeer grote overschatting van de werkelijke kosten als de ratio van de gemeenschappelijke bestelkosten ten opzichte van de gemiddelde additionele bestelkosten groot is. (Zie hoofdstuk 6).

   - SILVER, E.A., A control system for coordinated inventory replenishment, *International Journal of Production Research* 12, 1974, 647-671.

5. Beschouw het model in hoofdstuk 8. Een alternatieve bestelstrategie is om aan het begin van iedere periode de economische voorraad aan te vullen tot een niveau S. De gemiddelde voorraad, bij een gegeven waarde van S, is dan gelijk aan

$$\frac{1}{C}\sum_{i=0}^{\infty}\omega_c(S-i)\,f_C(i)\,,$$

waarbij $f_C(i)$ volgt uit de volgende recursieve betrekking:

$$f_t(i) = p_t\phi_t(i) + (1-p_t)f_{t-1}(i)\ ,\ t = T,..,C.$$

(Zie voor de notatie hoofdstuk 8).

6. In deze tijd, waarin organisaties zich steeds meer bewust worden van de voordelen van het uitwisselen van informatie met afnemers en/of leveranciers, is het zinvol om modellen te ontwikkelen die deze voordelen kwantificeren. (Zie hoofdstuk 8).

7. Het gevaar van het gebruik van geautomatiseerde bestelsystemen is dat de gebruikers vaak niet op de hoogte zijn van de wijze waarop de parameters zijn ingesteld. Hierdoor wordt niet adequaat gereageerd op veranderingen in de onderliggende input-parameters.

   Voorbeeld: In geautomatiseerde bestelsystemen worden bestelpunten vaak uitgedrukt als de verwachte vraag gedurende een x-tal weken, waarbij x door de gebruiker moet worden ingevoerd. Stel nu dat de gemiddelde levertijd kan worden verkort door de invoering van EDI (Electronic Data Interchange) met de leverancier. In het algemeen zal de gebruiker de parameter x niet aanpassen, aangezien niet bekend is dat x is opgebouwd uit de levertijd en een veiligheidstijd. Dit heeft tot gevolg dat de invoering van EDI zal leiden tot een hoger voorraadniveau, terwijl het tegenovergestelde effect wordt beoogd.

8. In het kader van een meer toepassingsgerichte benadering door de wetenschap verdient het gebruik van heuristische oplossingsmethoden voor realistische problemen de voorkeur boven optimale oplossingsmethoden voor problemen die een sterk vereenvoudigde weergave vormen van de werkelijkheid.

9. Bij de keuze van de wijze van inbinden van het proefschrift (genaaid of lijmgebonden), waarbij een afweging wordt gemaakt tussen kosten en kwaliteit, dient de promovendus rekening te houden met het waarschijnlijk zeer beperkt aantal keren dat zijn boekwerk zal worden doorgebladerd.

10. Een jonge doctor feliciteren met zijn of haar afstuderen (hetgeen nogal eens voorkomt) is hetzelfde als een pas getrouwd stel geluk wensen met hun verloving.

11. Solliciteren naar een functie in de advies-sector is te vergelijken met het lonken naar een mooie vrouw, die wordt begeerd door een zeer groot aantal (ervaren en minder ervaren) knappe mannen.

COORDINATED REPLENISHMENT POLICIES

COORDINATED REPLENISHMENT POLICIES

Proefschrift

ter verkrijging van de graad van
doctor aan de Katholieke Universiteit Brabant,
op gezag van de rector magnificus,
prof. dr. L.F.W. de Klerk,
in het openbaar te verdedigen
ten overstaan van een door het college van dekanen
aangewezen commissie in de aula van de Universiteit
op vrijdag 8 oktober 1993 te 16.15 uur

door

Marcus Josephine Guilaume van Eijs

geboren te Sittard

druk: wibro dissertatiedrukkerij, helmond.

Promotor:

prof. dr. F.A. van der Duyn Schouten

Co-promotor:

dr. R.M.J. Heuts

VOORWOORD

Voorraadmanagement, of meer in het algemeen logistiek, wordt gezien als één van de cruciale factoren voor een succesvolle bedrijfsvoering in de jaren negentig. Kwantitatieve modellen kunnen als nuttig hulpmiddel dienen voor de vaak zeer complexe besluitvorming op dit gebied. In dit proefschrift worden enkele bestaande modellen geanalyseerd en worden enkele nieuwe modellen toegevoegd aan de literatuur over *gecoördineerde bestelstrategieën*. Op deze manier wordt getracht vanuit een besliskundige invalshoek een bijdrage te leveren aan het verbeteren van het voorraadbeheer in bedrijven en organisaties.

Dit proefschrift is het resultaat van het onderzoek dat ik heb verricht in de periode van oktober 1988 tot en met mei 1993 als Assistent in Opleiding bij de vakgroep Econometrie aan de Katholieke Universiteit Brabant. Graag wil ik enkele mensen bedanken die hieraan een bijdrage hebben geleverd.

In de eerste plaats bedank ik prof. dr. Frank van der Duyn Schouten en dr. Ruud Heuts voor hun stimulerende en intensieve begeleiding. Zij hebben elk op eigen wijze een belangrijk aandeel gehad in de totstandkoming van dit proefschrift. Ik ben me ervan bewust dat dit boek niet zou zijn geschreven zonder hun deskundige adviezen ten aanzien van de aanpak en uitvoering van het onderzoek.

Prof. dr. A.G. de Kok, prof. dr. J.P.C. Kleijnen, prof. dr. M. Lambrecht en prof. dr. B.B. van der Genugten wil ik bedanken voor het feit dat zij bereid waren om in mijn promotie-commissie zitting te nemen. In het bijzonder wil ik prof. dr. Jack Kleijnen bedanken voor de samenwerking die heeft geleid tot het onderzoek dat wordt beschreven in hoofdstuk 2.

Met veel collega's heb ik gesproken over mijn onderzoekswerk. De interessante discussies die ik met Erik van der Sluis van de Universiteit van Amsterdam heb gevoerd over ons gemeenschappelijke onderzoeksonderwerp wil ik hierbij niet onvermeld laten. I also want to thank prof. Edward Silver from the University of Calgary for the encouraging and pleasant discussions we had in Budapest and Amsterdam.

De collega's van de vakgroep Econometrie, en met name de collega-AIO's, wil ik bedanken voor de prettige werksfeer. In dit kader gaat een bijzonder woord van dank uit naar mijn kamergenoot Rob van der Mei. De plezierige omgang, evenals de vele discussies, al dan niet over het proefschrift, hebben in grote mate bijgedragen tot het vruchtbare werkklimaat waarin dit proefschrift kon worden geschreven.

Ik dank Rob tevens voor zijn opmerkingen ten aanzien van eerdere versies van dit proefschrift. Daarnaast hebben Michel van Eijs, Toine Slaats en Arjan Westerhof delen van het proefschrift gescreend. Ik ben hen hiervoor zeer erkentelijk.

Mijn ouders, schoonouders, vrienden en kennissen wil ik hartelijk dankzeggen voor hun meeleven gedurende mijn promotie-onderzoek. Hun belangstelling werkte immer zeer motiverend.

Tenslotte wil ik Irene bedanken voor alle steun die ik door de jaren heen van haar heb mogen ontvangen. Ze heeft me alle tijd en ruimte gegeven die, vooral in de afrondingsfase, nodig was voor de voltooiing van dit proefschrift.

Tilburg, augustus 1993 Marc van Eijs

CONTENTS

PART II

CHAPTER 1

OVERVIEW

This chapter gives an overview of the thesis. Inventory systems are divided into independent demand systems and dependent demand systems. We focus on the design and analysis of coordinated replenishment systems within the class of independent demand systems. After a detailed review of existing coordinated replenishment policies, the contents of this thesis is summarized.

1.1 Introduction

Inventories are held in every organisation. They form a buffer against discrepancy between supply and demand processes. The cost of holding and control of inventories represent a considerable amount of investment and operating costs. Efficient inventory control, which tries to minimize inventory related costs while maintaining a high customer service, is therefore an important factor for the competitive strength of an organisation.

Due to the large number of individual items (hundreds or thousands), the diverse collection of relevant factors (e.g. demand patterns, different modes of shipment from suppliers or delivery to customers), and constraints (e.g. budget limitations, vendor restrictions, and customer service levels), the use of quantitative decision support models in an inventory system is to be recommended.

During the last three decades many Operations Research models have been proposed to tackle inventory management problems. Nevertheless, only a few of them are used in business. The main reason for this gap between inventory theory and practice is the modelling approach in the past. In particular in the sixties and seventies, Operations Research models focused on getting an optimal solution to a mathematically interesting, but unrealistic formulation of an inventory management problem. The problem formulations were characterized by an extreme simplification of the original problem (e.g. by several model assumptions or by considering only small problems). The scepticism with respect to the applicability of Operations Research techniques for real-life situations originates mainly from this type of models.

However, in the last decade, there is a trend in the literature towards the analysis of complex inventory models, which give a more accurate representation of the real-life problem. In many of these situations exact solutions of the relevant models are impossible. Moreover, when an exact solution can be obtained, it quite often results in complex decision rules which are hard to implement in practical situations.

In this study we therefore focus attention on the design and analysis of control rules which are on one hand good enough (in the sense that they are close to the optimal control rule), and on the other hand are easy to implement. Ideas for these heuristic decision rules will be based on insights obtained from the analysis of models, using techniques of Operations Research.

One of the simplifying assumptions which is made in most inventory management models is the independency of decisions made for different items. The main part of inventory management literature considers independent replenishment of a single item, whereas joint replenishments are common practice in real-life procurement processes. Coordination of replenishments of a group of items makes sense when these items are purchased from the same supplier or share the same mode of transportation. Such coordination may lead to a reduction in the inventory related cost, due to reduced ordering costs, reduced freight rates, reduced handling costs, quantity discounts or improvement of the implementation of stock control.

In this thesis we analyze and compare some existing models and add some new models for coordinated control in several practical situations. The objective of the study is to support managerial decision making in complex situations, using Operations Research models.

1.2 Background of inventory systems

Inventory control is a subarea of logistics, which can be defined as the total range of activities concerned with the movement of products, including information and control systems. Due to the changing market and industrial environment, logistics has become one of the critical success factors for the nineties. As opposed to the sixties and the early seventies, a poor management of the flow of goods can spell the difference between success and failure in the market. Some general trends are the diversification of customer needs, shorter product life cycles (due to the rapidly changing technology and requirements), increased level of automation, and globalisation of production and distribution.

Faced by these trends, companies need to be more flexible and need to provide a higher customer service. On the other hand, to remain competitive, operating costs have to be decreased. It is clear that the external goals (higher flexibility and customer service) conflict with the internal goals (lower operating cost). Inventory planning and control is one of the logistics elements that copes with these conflicting goals. An inventory system provides the organizational structure and the operating policies for maintaining and controlling goods to be stocked. The inventory system gives answers to the two fundamental questions in inventory control: (i) when should orders be placed (timing of orders), and (ii) how much should be ordered (determination of order quantities) ?

Inventory control systems can be classified in a number of ways. One classification differentiates between *dependent demand systems* and *independent demand systems*.

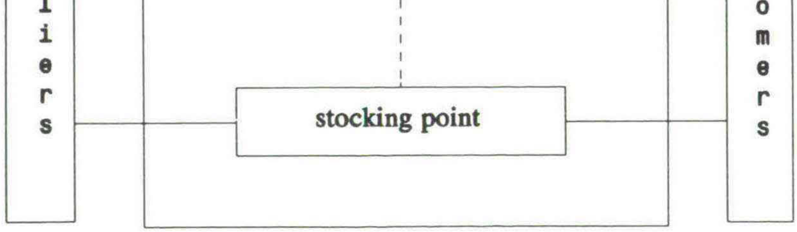
Dependent demand systems

These systems assume that the demand for an item is directly related to the demand for other items. Examples of items with dependent demand are substitutable items (when a particular item is not in stock, the customer may be willing to accept a substitute) or complementary items (when the customer may not accept one item without the other). However, dependent demand particularly occurs among items at different levels in the goods flow in an assembly or component industry (e.g. in a car assembly plant, the demand for doors and wheels are both directly related to the demand for finished cars). Material Requirements Planning (MRP I) has been introduced for an efficient coordination of materials and sub-assemblies in such situations. MRP I, which is most suited in a low volume batching situation, assumes that demands for finished goods can be forecasted accurately. The demand for the raw materials and sub-assemblies can then be related to the planned production of finished goods. Manufacturing Resource Planning (also referred to as MRP II or Closed loop MRP), which is an extension of MRP I, includes a feedback about capacities needed by all operations to ensure that there is enough to meet the production plans. The reader is referred to Orlicky (1975) or Vollmann et al. (1988) for more details on MRP systems. Another control system which is suited to handle capacity restrictions in a manufacturing/assembly environment, is Optimized Production Technology (OPT) or Theory of Constraints (see Goldratt and Fox (1986)). This theory focuses on the determination and control of bottle necks in a production line. Considerable interest has been shown in an alternative approach, which is known as the Just in Time (JIT) philosophy (see e.g. Hall (1983)). The aim of JIT is to minimize stocks of materials by having them arrive just when they are needed in the right quantity and quality. Each working centre in the production line produces only what its following working centre needs. The production at each working centre is controlled by a simple manual information system, which is called Kanban. JIT is appropriate in a high volume, repetitive manufacturing environment.

Independent demand systems

These systems assume that the demand for an item is independent of the demand for any other item. Then the aggregate demand for an item is made up of many independent demands of individual customers for that item. In these circumstances the only reasonable approach to forecast aggregate future demand per item is to project historic trends. Inventory control is then based on quantitative models which relate demand, cost, and other variables to find optimal values for order timing and order quantities. This type of control is often referred to as Statistical Inventory Control (SIC).

In this thesis attention is focused on independent demand systems. The main aim of this kind of control systems is to reduce the inventory related costs, while maintaining a high customer service. A typical representation of an independent demand inventory system is depicted in Figure 1.1. Applications of such systems can specifically be found in



---- : information flow, —— : goods flow

Figure 1.1 Representation of independent demand system a non-production environment, such as distribution organisations, retailers, wholesalers as well as service industries.

There are two major flows in the system: a *goods flow* and an *information flow*. Typically, stocks are held at an intermediate stocking point between external suppliers and external customers. (The external customers could also be considered as another stocking point in a distribution network, or as another internal department in a production system, but, in general, the demand will not be independent in such cases.)

The inventory system, which is concerned with the day-to-day operational decisions of when to order and how much to order, is supported by a flow of information. The developments in information technology have contributed to the design of automated information systems. In the beginning, these systems were used for registration purposes only. However, due to the increased complexity of planning and control, the systems have been extended to Management Information Systems, which enable the manager to control the logistics chain and to respond more flexibly to changing market conditions. A recent trend in the area of information systems is the automated exchange of information with suppliers or customers. (Electronic Data Interchange (EDI) is an example of such an exchange of information.) A possible categorization of the information flow is shown in Table 1.1.

Table 1.1 Categorization of information

Category	Examples
1. Constraints	1. desired customer service level, budget constraints, limited production capacity, limited storage space
2. Costs	2. purchasing costs, ordering costs, backordering costs, transportation costs, handling costs, system costs
3. Inputs from operations planning	3. stock on hand, stock on order, backorders, forecasts, demand rates, lead times, discount opportunities
4. Decision support models	4. periodic review models, reorder point models, hybrid models

Decision support models also contribute to the flow of information. They provide a way of analyzing data, and they give some insight into the consequences of decisions to be made. The inventory models use information on the inventory system (such as ordering policies, costs, demands, service levels), and provide a proposal on the order timing and order quantities. The decision maker checks whether the proposal is acceptable with respect to constraints which are not incorporated in the model or some specific information about future demands or deliveries, and then issues the order.

An additional remark has to be made with respect to the measurement of the cost factors which are used in the inventory models. The guiding principle for identifying these costs is that they have to be *incremental* (or marginal) costs (i.e. the extra cost of placing one extra order or stocking one extra unit). Accounting costs, which are primarily developed for other purposes, are usually inappropriate for inventory decision making purposes. Shortage related costs (backordering costs or costs of lost sales) are particularly difficult to measure. This has led to the frequent use of service level constraints, which only implicitly specify a shortage cost.

There are various types of *ordering policies* for independent demand systems. A commonly used classification distinguishes between periodic review policies and reorder point policies. The key quantity in deciding when and how much to order is the *inventory position* of an item, which equals the stock on hand plus the stock on order minus the backorders.

Periodic review policies

A periodic review policy places orders at regular intervals of time (e.g. every Monday). The order quantity is a variable number of units which is enough to raise the inventory position up to a specified level.

Reorder point policies

Under a reorder point policy, the stocks are monitored after every demand event. An order is placed as soon as the inventory position drops to or below a given reorder point. Usually, the order quantity is a fixed quantity (for example the economic order quantity or a specified quantity such as a truck load or a standard packaging size). This type of policy is also referred to as a fixed order size policy or a continuous review policy.

Each approach has certain advantages in specific circumstances. The main benefit of a periodic ordering strategy is the low system and control cost. There is a regular routine of stock checking, order placing, and order receiving. The reorder point policy, however, triggers orders at irregular points in time and it requires an on-line registration system, which yields higher system and control cost. Another advantage of periodic review policies is the ease of combining orders of several items into a single order. This gives larger orders, which might lead to unit-price discounts or lower freight rates. On the other hand, a continuous review policy reacts more accurately to the actual inventory position. As a consequence, it allows lower safety stocks and, hence, lower holding cost.

The benefits of these two types of policy have resulted in several hybrid policies such as periodic review policies with reorder points or reorder point policies with variable order quantities. The choice of the policy depends on the situation and should largely be a management decision. In general, one may say that reorder point policies are better suited to low, irregular demand for relatively expensive items, whereas periodic review policies are preferred for high, regular demand of low value items.

Inventory models

Hundreds of models have been developed in the last three decades to support managers in the choice of an appropriate ordering policy for a certain inventory problem. Each standard text book on inventory models starts with the analysis of the Economic Order Quantity (EOQ) model. This classic model determines the optimal order size, which minimizes the total relevant cost for a *single* item, under a set of simplifying assumptions. The model assumes that demand is known exactly, continuous, and constant over time. All costs are known and constant; the unit purchasing cost and the ordering cost do not vary with the order quantity. Further, it is assumed that shortages are not allowed and lead time is negligible. In this case, the optimal order quantity equals

$$EOQ := \sqrt{\frac{2Db}{h}}\,, \tag{1.1}$$

where D denotes the demand per time unit, b denotes the ordering cost per order, and h denotes the holding cost per unit per time unit.

Despite of the unrealistic assumptions, the EOQ model is widely used in practice because of its simplicity and its robust nature. After the derivation of the EOQ formula (which is also referred to as Wilson's lot-size formula or Camp's formula), other models tried to remove some of the assumptions of the standard model. The models considered range from an EOQ model with quantity discounts on the unit purchasing cost to very complex stochastic multi-item multi-echelon inventory models with capacity constraints. It is not our intention to give an exhaustive description of all these models. An excellent overview of inventory management literature until the early eighties is given in Silver (1982).

1.3 Coordinated replenishment systems

The vast majority of the inventory literature is concerned with the control of individual items. These systems treat the inventory control of each item in isolation from all other items. However, there are many situations where, although demand for each item is independent, it is much more natural to consider interactions among items. We discuss some examples.

- *Interaction by limited capacity*
  It often occurs that there are constraints on the inventory system, such as limited storage place or a maximum acceptable investment in stock. Such a constraint can be incorporated explicitly into the model formulation. However, it is also possible to ignore the interaction in the model, and to adapt the ordering decisions (a posteriori) when the constraint is violated by the solution of the single-item models.

- *Interaction by multiple stocking points*
  Frequently, inventories are kept at more than one location, for example, at a central warehouse and at a number of retail outlets. It is clear that decisions concerning the same item at different locations should not be made independently. In case the demand of the final customers can be forecasted quite accurately, the inventories within the distribution system might be controlled by the principles of Distribution Requirements Planning (DRP), which is in fact a natural extension of MRP to multi-echelon distribution systems (see e.g. Martin (1983)). Other inventory systems which account for multiple stocking points, are the Base Stock control systems, which are based on the concept of echelon stocks which has been introduced by Clark and Scarf. (For a detailed list of references on this topic, we refer to Schwarz (1981).)

- *Interaction by joint cost structure*
  In many situations, it makes sense to coordinate the replenishments of several items. For example, when several items are transported with the same mode of transport, transportation costs may be reduced by combining orders for these items; when several

items share a common supplier, discounts may be offered on the total dollar value of the order, which encourages joint replenishment of a group of items. A reduction in ordering cost may also be possible because several items are processed under a single order.

Except for Chapter 8, the thesis will be devoted to multi-item problems with the latter kind of interaction. We distinguish two classes of joint cost structures: *joint ordering cost structures* and *joint discount structures*.

Joint ordering cost structures

In the literature on coordinated replenishment systems, the cost-effectiveness of combined orders is mostly modeled by a so-called joint ordering cost structure, where a joint ordering cost is incurred for any order and an individual ordering cost is incurred for each item included in the replenishment. In the literature, the joint ordering cost and the individual ordering cost are frequently referred to as the *major set-up cost* and the *minor set-up cost*, respectively.

Let S denote the set of items in the order, let A denote the joint ordering cost, and a_i the individual ordering cost of item i. Then the total ordering cost, K(S), equals

$$K(S) = A + \sum_{i \in S} a_i \ . \tag{1.2}$$

The joint ordering cost can be considered either as the fixed cost of placing an order, independent of the number of items (e.g. the cost of communication), or as the fixed transportation cost per ride. The individual ordering cost can be considered as the relatively minor cost of adding one more item to the order (e.g. the cost of administration or receiving and inspection of individual items).

Example 1.1

Recall that the classical EOQ model determines the order timing and order quantity in a simplified situation with constant demand and constant cost parameters. The EOQ model does not account for interaction by a joint ordering cost structure. Every item has its own ordering pattern and at every order moment of a particular item not only the item-dependent individual ordering cost is incurred, but also the joint ordering cost (since orders of different items coincide only by accident). Consider a family of two items, which has the demand and cost parameters given in Table 1.2.

Table 1.2 Input data for Example 1.1 (joint ordering cost = 10)

item	purchasing cost per unit	individual ordering cost	holding cost per unit per time unit	demand per time unit
1	15	5	1	100
2	10	3	1	200

Now, we might determine an individual ordering policy for each item, according to the EOQ model. In this case, item 1 orders 56 units every 0.56 time units and item 2 order 72 units every 0.36 time units (see formula (1.1)). We might also create a coordinated replenishment policy that orders both items jointly at every replenishment epoch. It can easily be shown that, in this case, it is optimal to order 35 units of item 1 and 70 units of item 2 every 0.35 time units. The total ordering and holding cost per time unit of the individual replenishment policy and the coordinated replenishment policy equal 126.88 and 103.92, respectively. The cost saving by the coordinated replenishment policy is caused by a considerable reduction in the ordering cost due to a decrease of the number of orders (the individual replenishment policy triggers 4.56 orders per time unit, while the coordinated replenishment policy triggers only 2.86 orders per time unit).

We have already mentioned some situations where coordinated replenishment systems make sense (common supplier, common transportation mode). There is a closely related production situation, where items share a common production facility and set-up costs depend on the sequencing of the items. The joint ordering cost can then be considered as the changeover cost associated with converting the facility from the production of a certain family to production of another family. The individual ordering cost represents the relatively minor cost of switching production within the same family. (An illustration of such a situation is the bottling of beer products. There is a major changeover cost when converting the production line from one quality of beer to another. In contrast, the cost of changing from one container type to another is minor.) In the literature about coordinated replenishment models, it is implicitly assumed that there are no constraints on the system. However, in this production situation, there is the additional complexity of a finite production or storage capacity. In this thesis we will usually consider coordinated replenishment systems in a non-production context.

Joint discount structures

The unit value of an item may depend upon the size of the replenishment. This may be a result of a supplier discount or it can come about through freight considerations (e.g. truck load versus less-than-truck load). Joint discount structures differ from joint ordering cost structures in the sense that the cost saving obtained from joint ordering depends on the total size of the combined order (expressed in dollars, quantities, volume, or weight).

When a group of items is ordered from the same supplier, a quantity discount may be offered if the total order is larger than a given threshold. Examples of such discount structures are the all-units discount structure or the incremental-units discount structure where a percentage discount is offered on the unit purchasing cost. Instead of ordering a large amount of a single item, it may be economical to combine orders of individual items to achieve the discount breakpoint. (This kind of joint consideration is also useful in case the supplier imposes a minimum size of the order.)

The cost of transport per unit delivered (the freight rate) will usually decline with an increasing size of the order. Depending on the delivery vehicle, discounts may be achieved by ordering quantities which correspond to a pallet load, a container load, or a truck load. This may often be accomplished by the simultaneous ordering of several items. In some cases, the transportation cost can be contained in the joint ordering cost structure (e.g. when there is a fixed price per ride). However, in several other cases, the freight rate schedule has to be considered explicitly.

Example 1.1 (continued)

Suppose that a discount of 5% is offered on the total order if the value exceeds 1200 dollar. Under the individual (EOQ) replenishment policy, the total dollar value is 840 if item 1 is ordered and 720 if item 2 is ordered. Hence, the discount will never be incurred. Under the coordinated replenishment policy, the joint order (with a value of 1225 dollar) always exceeds the threshold, so the purchasing cost per time unit is decreased by 175 dollar ($0.05 \cdot (100 \cdot 15 + 200 \cdot 10)$) compared with the individual policy. The discount would also be incurred by the individual replenishment policy if the order quantity for item 1 is increased to 80 units and that of item 2 to 120 units. However, in this case the ordering and holding cost per time unit of the individual policy is 36.50 dollar higher than for the coordinated replenishment policy.

1.4 Overview of ordering policies for coordinated replenishment systems

The main part of the literature on coordinated replenishment systems is devoted to models with a joint ordering cost structure. Most of these models have their counterparts in the single-item literature. However, the multi-item models are mathematically more involved; obtaining optimal solutions for these models is either impossible or it requires a huge computational effort. Consequently, several heuristic approaches have been suggested.

The first part of this section deals with deterministic coordinated replenishment models; the second part reviews the stochastic version of the problem.

1.4.1 Deterministic models

Deterministic models assume that demand rates, cost parameters, lead times and other problem parameters are known exactly over time. We distinguish between the constant demand case and the time-varying demand case.

Constant demand case

Coordinated replenishment models for the constant demand case are extensions of the classical EOQ model, which assumes independent control of items, to the multi-item situation with a joint ordering cost structure. Even in this situation, which considers the most simplified coordinated replenishment system, the cost-optimal policy can usually not be identified. Therefore, attention has been restricted to special classes of ordering policies, which are on one hand close to the (unknown) optimal policy, and on the other hand, can theoretically be analyzed and easy be implemented. In fact, the existing policies appear to fall into one of the following two classes (or a combination of both).

Direct grouping strategies

Each strategy in this class employs a fixed partition of the items into groups. Each time when an item is ordered, it is ordered jointly with the other members of its group. The time between two successive replenishments of a group is constant. No joint replenishment occurs between items that are assigned to different groups.

Indirect grouping strategies

Under this type of strategy, the items are also classified into several groups with a common order interval. Items in the same group are jointly replenished at constant time intervals. The order intervals of the groups are chosen as integer multiples of some basic cycle time. (A family replenishment is made at constant time intervals. The replenishment cycle of each item is an integer multiple of this basic cycle time. A group is (indirectly) formed by those items that have the same replenishment frequency.) So, in contrast to direct grouping strategies, joint replenishments of different groups occur at certain multiples of the basic cycle time.

A survey of approaches to calculate the optimal control parameters of indirect and direct grouping strategies is given in Chapter 2. The main part of the literature considers indirect grouping strategies. However, it can be shown (see e.g. Andres and Emmons (1976)) that equal time spacing between replenishments of different groups (as indirect grouping strategies have) does not necessarily give an optimal solution. Moreover, it may not even be optimal to space orders of individual items equally. (Note that both indirect grouping and direct grouping strategies have fixed replenishment cycles for the individual items.)

A special subclass of the class of indirect grouping strategies are the power-of-two policies under which the replenishment cycle of each group is a power-of-two multiple of some basic period (e.g. a week). Roundy (1985) has given an algorithm which guarantees a power-of-two policy with a cost which does not deviate more than 2% of the cost of the (unknown) optimal policy. This result is extended by Federgruen and Zheng (1992) to more general cost structures than the standard joint ordering cost structure.

Several combinations of indirect grouping and direct grouping strategies have been proposed. Among others, Chakravarty (1984) considers policies which partition the items into a number of groups and apply an indirect grouping strategy within each group. Another combination has been considered by Goyal and Soni (1986).

The coordinated replenishment problem with constant demand is also closely related to the economic lot scheduling problem (a multi-item single-machine scheduling problem) and the two-echelon distribution problem (a single-item distribution problem for a system consisting of a single warehouse and several retail outlets). The relationship with these problems is clearly shown in Graves (1979) for the case of indirect grouping strategies.

It is clear that the constant demand case is not a realistic description of real-life. Nevertheless, the coordinated replenishment models for constant demand serve as a key building block in decisions rules for more complicated situations.

Time-varying demand case

The dynamic coordinated replenishment problem is concerned with the lot sizing of several items over a given time horizon, when demands in each period are known, but may vary from one period to another. Dynamic multi-item lot size problems may be useful, for example, for the order timing of materials in a MRP schedule.

Several models have been suggested to obtain the optimal solution for this problem. These models may be regarded as generalizations of the single-item dynamic lot size model of Wagner and Whitin (1958). The Wagner-Whitin condition, which states that in the optimal solution a given item is ordered only if its inventory position is equal to the demand during the lead time, is still valid in the multi-item case.

For a T period problem, the method of Veinott (1969) considers 2^{T-1} patterns of ordering at least one product or none in each period. Given such a pattern of family replenishments, the optimal ordering schedule for each individual item can be obtained from the Wagner-Whitin algorithm. The optimal solution follows after enumeration over all possible patterns. Based on this idea, Erenguc (1988) developed a branch and bound algorithm which avoids consideration of all possible patterns.

Other optimal solution methods are based on a dynamic programming formulation. The methods of Zangwill (1966), Kao (1979), and Silver (1979) differ basically in the choice of the state variable. These dynamic programming algorithms have a computational complexity which is exponential in the number of items.

All these solution methods can be used only for small sized problems. For larger (practical) problems, several heuristics have been suggested. Most heuristics determine lot sizes on a period by period basis. These so-called *single-pass heuristics* are generalized versions of existing single-item heuristics. Lambrecht et al. (1979), Joneja (1990), and Iyogun (1991) give simple extensions of the Part-Period-Balancing heuristic, whereas heuristics provided by Silver (1976), Atkins and Iyogun (1988a), Joneja (1990), and Iyogun (1991) are based on the Silver-Meal heuristic for single-item problems. (Both single-item heuristics are described in Silver and Peterson (1985).) Joneja and Iyogun also provide worst case performance bounds of several single-pass heuristics.

A comparative study by Litjens and Smits (1992) of the coefficient method of Lambrecht et al., the cost covering heuristic of Joneja, and the extended Silver-Meal heuristic of Silver showed that the heuristic of Lambrecht et al. gives the best results over a wide range of problems.

Multi-pass heuristics have been suggested by Kao (1979) and Van der Sluis (1991). Joneja has proven that the heuristic of Kao can perform arbitrarily bad. However, Van der

Sluis concluded from his computational results that his multi-pass heuristic outperforms the single-pass heuristics considerably.

1.4.2 Stochastic models

In stochastic models, demand behaviour is the major source of uncertainty. Usually, it is assumed that demand per time unit is not known exactly, but follows a known probability distribution (e.g. normal, gamma, or Poisson). Stochastic demand complicates the design of a (close to) optimal decision rule for a coordinated replenishment system. Below, we present an overview of existing replenishment policies which cope with this difficult problem. As in Section 1.2, the policies are classified into periodic review policies and reorder point policies.

Periodic review policies

Joint ordering costs, which do not depend on the number of items in the order, may be allocated to several items when these items are jointly replenished. Joint replenishments can be easily realized under a periodic review policy by synchronizing the review intervals of several items. The following periodic coordinated replenishment policies, which are widely used (implicitly or explicitly) in practice, are considered in the inventory management literature.

- *(R, S_i) policy*

  Under this simplest type of policy, the inventory position of item i is raised up to the order-up-to level S_i every R periods. Here, R is a common review period for all items in the family.

- *(R_i, S_i) policy*

  The former policy has a common review period for all items. However, in some cases it may be uneconomical to include all items in every replenishment. The (R_i, S_i) policy eliminates this inefficiency by allowing item-dependent review periods. To achieve coordination, the review period R_i of item i is chosen as an integer multiple of a basic period. (Note that this policy is in fact a generalisation of the indirect grouping policy to the stochastic demand case.)

- *(R_j, S_{ij}) policy*

  This policy is a generalisation of direct grouping strategies to stochastic demand: the items are partitioned into a number of groups with a common review interval. The inventory position of each item i belonging to group j is raised up to S_{ij} every R_j periods. No joint replenishment occurs among items that are assigned to different groups.

In the literature there exist several procedures to calculate the optimal order-up-to level of an item corresponding to a given review period. (See e.g. Silver and Peterson (1985) or Hadley and Whitin (1969). A very general approach has been suggested by De Kok (1991a).) In practice, the review periods are frequently determined by external factors, such as the delivery schedule, or they are based on arbitrary rules. (A typical decision rule would be based on an ABC analysis; e.g. order A-items every week, order B-items every 12 weeks, and order C-items once in a year.) Inventory models may support the managers in setting the cost-optimal review periods. See e.g. Atkins and Iyogun (1988b), Chakravarty (1986), Naddor (1975), and Chakravarty and Martin (1988).

Reorder point policies

Although periodic review policies are usually used in practice, the literature has tended to concentrate on reorder point policies with continuous review. In particular, attention has been focused on the so-called can-order policies or (S_i, c_i, s_i) policies.

- *(S_i, c_i, s_i) policy*

  The can-order policy is characterized by a set of three parameters for each item i, namely (S_i, c_i, s_i). Whenever the inventory position of any item i drops to or below its must-order point s_i, a family replenishment is made. All the other items j within that same family with an inventory position less than or equal to their can-order point c_j are included in this replenishment. The inventory position of each item k in the replenishment is then raised to the order-up-to level S_k.

  Although the control mechanism of the can-order policy is very easy, it is difficult to determine the optimal control parameters. Silver (1974, 1981), Thompstone and Silver (1975), and Federgruen et al. (1984) have proposed heuristic algorithms to find the parameters of the optimal can-order strategy for the case of (compound) Poisson demands. These iterative procedures are based on a decomposition of the multi-item problem into a number of appropriately chosen single-item problems. This approach will be analyzed in Chapter 6.

- *(Q_i, c_i, s_i) policy*
  An obvious alternative for the can-order policy, with variable order quantities, is a similar policy with fixed order quantities Q_i, such as a pallet size or a container size.

- *(Q, S_i) policy*
  The can-order policy triggers an order whenever any item drops to or below its must-order point. Under a (Q,S_i) policy, the order timing is based on the combined inventory position of *all* items in the family. Whenever this combined inventory position falls to the group reorder point, an order is placed to raise the inventory position of all items i to their order-up-to level S_i. Under unit demand sizes, the group reorder point is reached whenever the combined usage since the last order reaches Q. (See Pantumsinchai (1992) for a procedure to calculate the optimal parameters of this inventory control rule in case of Poisson demands.)

Combinations of periodic review and reorder point policies
There are also several combinations of periodic review policies and reorder point policies.

- *(R, s_i, S_i) policy and (R, s_i, Q_i) policy*
  Under these inventory control rules, the inventory position of each item is checked every R periods (R is equal for all items), and item i is included in the family replenishment if its inventory position is at or below the reorder point s_i. The order quantity is either variable (raising the inventory position to S_i) or a fixed number of units (Q_i). (See e.g. Naddor (1975) or De Kok (1991b) for the calculation of the optimal parameters of these policies.)

- *Service point policy*
  The service point policy is a periodic review policy which uses a group reorder point. The policy works as follows: at a review time, the available stock for each item in the family is checked to determine the expected shortage in the current replenishment cycle if the order is delayed for one review time. The total expected shortage for the entire family is compared with the allowed shortage which is calculated from the required service level. A family order is triggered whenever the expected shortage is more than the allowed shortage; otherwise ordering is delayed for at least one review period. The

order quantity is determined from an analysis of the available stocks and the inventory related costs.
This policy has been implemented in IBM's system IMPACT (see IBM (1971)). Improvements of the original service point policy have been suggested by Kleijnen and Rens (1978) and Carlson and Miltenburg (1988).

- *(σ , S_i) policy*
The (σ , S_i) policy is another periodic review system where the order timing depends on the inventory position of all items. When, at a review time, the combination of inventory positions is in an order region, which is characterized by σ, then the inventory of all items i is brought to S_i. (See e.g. Sivazlian and Wei (1990).)

1.5 Overview of the thesis

This thesis is divided into three parts:

- Part I deals with deterministic coordinated replenishment problems with a joint ordering cost structure.
- Part II considers stochastic coordinated replenishment problems with different types of joint cost structures.
- Part III describes an approach to quantify the value of information in inventory management.

Part I considers the constant demand case of the joint replenishment problem. As we mentioned in the previous section, indirect grouping strategies and direct grouping strategies are close to optimal ordering policies which account for the joint ordering cost structure. One might conjecture that indirect grouping strategies outperform direct grouping strategies for high joint ordering costs, because they explicitly account for possible savings due to the synchronization of group replenishment cycles. On the other hand, indirect grouping strategies are less flexible in setting replenishment cycles, since these cycles are restricted to integer multiples of the basic cycle time. The performance of both strategies, which is measured as the percentage cost saving relative to an independent strategy, is compared in Chapter 2. It appears that the performance depends on the ordering cost ratio (i.e. the ratio of the joint ordering cost and the average individual ordering cost) and the number of items. Further, we conclude that indirect grouping strategies, in general, outperform direct grouping strategies, except for situations where the ordering cost ratio is very small (less than 0.50) or very large (more than 50).

In the literature, a solution procedure developed by Goyal (1974) is used to find the global optimum within the class of indirect grouping strategies. In Chapter 3 we show that Goyal's algorithm does not always lead to the optimal indirect grouping strategy. We propose a simple correction. Numerical investigations show that the correction is necessary only when the ordering cost ratio is very small (<0.2).

Part II describes some stochastic coordinated replenishment models. The literature on this type of models has almost exclusively been confined to settings where economies of scale in joint replenishments are restricted to reduced joint ordering costs. However, in practice several other incentives for coordinated control exist, such as quantity discounts and freight rate discounts.

Chapter 4 and 5 deal with continuous review multi-item inventory systems which account for both joint ordering costs and unit-price quantity discounts. The class of can-order policies has already been introduced. Savings of using can-order policies are due to reduced joint ordering costs. However, these strategies, which are simple implemented in practice, do not take discount possibilities into account. In Chapter 4 we propose a hierarchical policy that incorporates discounts into the framework of can-order policies. This replenishment system is referred to as the CAN^+ system.

To evaluate the performance of this system, we compare it with a coordinated replenishment system which has been developed by Miltenburg and Silver. We may conclude that, based on the underlying demand processes, Miltenburg and Silver's system is preferred for fast movers, whereas in case of erratic demand the CAN^+ system is preferred. In order to make a numerical comparison between both systems, Miltenburg and Silver's system is adapted for Poisson demands. The numerical results show that the performance of the CAN^+ system is approximately equal to that of Miltenburg and Silver's system as far as the controllable costs are concerned. The modification and the comparison are discussed in Chapter 5.

Chapter 6 investigates the determination of the optimal can-order policy. Traditionally, the optimal control parameters are determined by an iterative procedure which relies on a decomposition approach. It is shown that this method gives inaccurate results when the ordering cost ratio is large, because the underlying assumption for the decomposition in not valid in this case. (In some cases the model overestimates the real cost by more than 20%.) For problems with a large ordering cost ratio, we restrict attention to a subclass of can-order policies, and develop a solution procedure to determine the control parameters of this special policy.

When several items share the same transportation facility, coordination of orders may lead to reduced freight rates. Chapter 7 considers a multi-item inventory system with transportation economies when ordering a full-container load instead of a less-than-container load for transportation from overseas. We propose a periodic review policy which incorporates the special transportation cost schedule into the analysis of the order composition of a family of items. Some numerical examples show that the total cost can be substantially decreased (up to 20%) in case ordering and transportation planning are integrated.

The topic of *Part III*, quantification of the value of information, lies somewhat beyond the scope of the models in Part I and Part II. Under pressure of the Just in Time philosophy, there is a trend towards closer relations between retailers (or manufacturers) and their suppliers. Based on these developments, there is an increasing awareness that exchange of information in the logistics process can be beneficial to all parties involved. However, quantification of the benefits of information interchange is usually not easy.

Chapter 8 deals with the situation where a supplier, who produces on order in fixed production cycles, provides information about the status of upcoming production runs. Such information enables the decision makers to improve their inventory control. We present a fixed order quantity policy with a set of reorder points corresponding to the prospective lead times, depending on whether the next production run is filled. A Markov model that analyses such a type of control rule, is used to quantify the value of information. The numerical examples show that the approach may lead to considerable cost savings compared with the traditional approach that uses only one single reorder point, based on a two-moments approximation of the demand during lead time and review time.

Summarizing,

- Chapter 2 compares the performance of indirect grouping and direct grouping strategies.
- Chapter 3 gives a correction of Goyal's algorithm which calculates the control parameters of the optimal indirect grouping strategy.
- Chapter 4 describes a new class of replenishment policies which incorporates the evaluation of discount-opportunities within the framework of can-order policies.
- Chapter 5 compares the performance of this class of policies with the performance of the inventory control system developed by Miltenburg and Silver.

- Chapter 6 indicates that the traditional method to determine the optimal can-order policy performs poorly in case of a large ordering cost ratio, and develops a procedure to calculate the parameters of an appropriate special can-order policy.
- Chapter 7 describes a new class of coordinated replenishment policies with joint determination of ordering and transportation decisions.
- Chapter 8 describes an approach to quantify the value of information about the supplier production run status in inventory management.

As we have mentioned before, the goal of this study is to support managerial decision making, by using quantitative models. In the *design* of the replenishment policies, the emphasis is on the applicability of the decision rules in practical situations. The ordering rules have to be understood by the decision maker and the algorithms should be implementable into a decision support system on a PC. The *analysis* of the models is based on a combination of mathematical reasoning, application of Operations Research techniques, and heuristic thinking. In particular, the models in Parts II and III are mainly based on the theory of Markov decision processes. Further, simulation is an indispensable tool for the evaluation of the heuristic decisions rules.

The subsequent chapters are all based on papers, which have been submitted to international journals. The following (combinations of) chapters can be read without having to go through the entire manuscript:

- Chapter 2 and 3,
- Chapter 4, 5, and 6,
- Chapter 7,
- Chapter 8.

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CHAPTER 2

ANALYSIS AND COMPARISON OF INDIRECT AND DIRECT GROUPING STRATEGIES

Multi-item inventory systems with joint replenishment costs are studied for constant deterministic demand. Two different types of strategies are distinguished: direct grouping strategies and indirect grouping strategies. For these types of strategies different heuristics are reviewed. The performance of both joint replenishment strategies is compared. The input-output behaviour of several simulation experiments is summarized by regression analysis.

2.1 Introduction

In many practical situations it makes sense to coordinate replenishments of individual items. If several items are purchased from the same supplier, the fixed ordering or transportation cost can be shared or group discounts can be achieved by replenishing two or more items jointly. Joint replenishments may also be attractive if a group of items use the same machine.

A realistic way to model the cost effectiveness is by the joint ordering cost structure, where a joint ordering cost is incurred for any order and an individual ordering cost is incurred for each item included in the replenishment. The joint ordering cost can be considered either as the fixed cost of placing an order, independent of the number of items in the order, or as the changeover cost associated with converting a production facility from the production of some other family to production within the family of interest. The individual ordering cost can be considered either as the relatively minor cost of adding one more item to the order or as the cost of switching to production of another item within the same family. (In a production context, the joint ordering cost and the

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individual ordering cost are usually referred to as major set-up cost and minor set-up cost, respectively.) We refer to Chapter 1 for a more detailed introduction to the problem.

In case of constant demand, the replenishment strategies for a family of coordinated items can be classified into two classes, which will be called *indirect grouping strategies* and *direct grouping strategies*. The strategies from both classes are based on fixed replenishment cycles for each individual item. (The replenishment cycle of an item is the time between two successive replenishments.) A *group* is defined as the set of those items that have the same replenishment cycle. Consequently, items of the same group are jointly replenished.

Under an indirect grouping strategy, a family replenishment is made at constant time intervals. The replenishment cycle of each item is an integer multiple of this basic cycle time. The problem is to determine the basic cycle time and the replenishment frequencies of all items, based on the ordering and holding cost. A group is then (indirectly) formed by those items that have the same replenishment frequency. In the last two decades several authors have studied this kind of joint replenishment problem. For extended reviews we refer to Kaspi and Rosenblatt (1985) and Goyal and Satir (1989).

A different type of strategy, which is not mentioned in these surveys, is a direct grouping strategy. Here, the replenishment cycles of the groups are not an integer multiple of a basic cycle, so the family replenishments are not equally spaced. Extra savings which may be obtained by synchronizing the replenishment cycles of the groups are ignored. In this case the problem is to determine (directly) the composition (and the replenishment cycles) of a variable number of groups in such a way that the sum of the ordering and holding cost of the items in the family is as low as possible.

Example 2.1

Consider a family of three items (a,b,c). Denote the basic cycle time of an indirect grouping strategy by T and the number of basic cycles between two orders of item i by k_i. Let $k_a=1$, $k_b=2$, $k_c=2$, and T=1.5. So, two groups have been formed. Group 1 (consisting of item a) has a replenishment cycle of 1.5 periods, and group 2 (item b and c) has a replenishment cycle of 3.0 periods. Consequently, group 1 and 2 are jointly replenished at time 0, 3, 6, 9, .. . An alternative strategy is a direct grouping strategy where two groups are formed: group 1, consisting of item a, with a replenishment cycle of 1.5 and group 2, consisting of item b and c, with a replenishment cycle of 3.1. Note that group 1 and 2 are never jointly replenished under the direct grouping strategy.

Another class of strategies which permits unequal time spacing between joint replenishments has been suggested by Goyal and Soni (1986). They provide an extension of the class of indirect grouping strategies by permitting up to three parallel basic cycles. Denote the basic cycle of the *j* th parallel cycle by T_j, then they set $T_1 = T$, $T_2 = 3 \cdot T$, and $T_3 = 5 \cdot T$, where T is a decision variable. Within each cycle the replenishments are undertaken at equal time intervals. As a consequence of permitting parallel cycles, some items may be switched from one parallel cycle to the other, which results in some feasible non-integer values of k_i. However, in the rest of this chapter, attention will be restricted to the class of indirect and direct grouping strategies.

One might conjecture that indirect grouping strategies outperform direct grouping strategies for high joint ordering costs, because not only items within a group, but even different groups are jointly replenished when using an indirect grouping strategy. However, indirect grouping strategies are less flexible in setting replenishment cycles, since these cycles are restricted to integer multiples of the basic cycle time. One can imagine that direct grouping strategies outperform indirect grouping strategies when the savings from coordination are small (low joint ordering costs). To the best of our knowledge, a comparison between the class of indirect grouping and direct grouping strategies has never been made. The purpose of the comparative study is twofold: first, to find out whether there is a threshold value of the joint ordering cost above which it makes sense to use an indirect grouping strategy; secondly to determine the effect of some factors in the performance evaluation of joint replenishment strategies. Performance is measured as the percentage cost saving when a joint replenishment strategy is used instead of an independent strategy.

This chapter is organised as follows: Section 2.2 discusses some solution procedures for indirect and direct grouping. Section 2.3 describes the experimental design and simulation results of the comparative study. The conclusions are presented in Section 2.4.

2.2 Solution procedures for indirect and direct grouping strategies

The joint replenishment problem can be described as follows: N items are purchased from the same supplier. When an order for one or more items of the family is placed a joint ordering cost is incurred, regardless of which items are ordered. In addition, an individual ordering cost is charged for each particular item which is included in the replenishment.

The demand for individual items is known and constant over an infinite horizon. Stock-outs are not allowed. The entire order quantity of an item is delivered at one time after a constant lead time. There are no discounts available on the unit purchasing cost. The objective is to minimize the sum of the ordering and holding cost per time unit over an infinite horizon. (Note that these assumptions are the same as those for the classical economic order quantity (EOQ) model, except for the joint ordering cost.)

The following notation will be used:

N : number of items in the family;
A : joint ordering cost;
a_i : individual ordering cost for item i (i=1,..,N);
h_i : holding cost per unit per unit time for item i (i=1,..,N);
D_i : demand per unit time for item i (i=1,..,N).

2.2.1 *Indirect grouping strategies*

An indirect grouping strategy is characterized by the following set of parameters:

T : basic cycle time (the time between two successive family replenishments);
k_i : number of basic cycles between two successive replenishments of item i (i=1,..,N).

The total average cost per time unit for an arbitrary indirect grouping strategy with parameters $(T;k_1,..,k_N)$ is given by

$$TRC(T;k_1,..,k_N) = \frac{1}{T}\,(A+\sum_{i=1}^{N}\frac{a_i}{k_i}) + \frac{T}{2}\sum_{i=1}^{N} D_i h_i k_i\ . \tag{2.1}$$

For a known vector $(k_1,..,k_N)$, the optimal value $T^*(k_1,..,k_N)$ of the basic cycle time equals

$$T^*(k_1,..,k_N) = \sqrt{\frac{2\,(A+\sum_{i=1}^{N}\frac{a_i}{k_i})}{\sum_{i=1}^{N} D_i h_i k_i}}\ . \tag{2.2}$$

On the other hand, given the value of T, the variable cost of item i is minimised by selecting the integer $k_i^*(T)$ which satisfies

$$k_i^*(T)(k_i^*(T)-1) < \frac{2a_i / D_i h_i}{T^2} \leq k_i^*(T)(k_i^*(T)+1) \ . \tag{2.3}$$

Hence, it is easy to derive the optimal T^* given the vector $(k_1, .., k_N)$, or the optimal k_i^* given T. However, T^* can not be determined without knowing k_i^*, and vice versa. Several authors have addressed this problem. Only one of them (Goyal (1974a)) presented an algorithm that provides the global optimum (under the assumption that the actual family replenishments are equally spaced). We refer to Chapter 3 for a detailed discussion of this optimal solution method.

Silver (1976) has developed a simple heuristic procedure to calculate the parameters of the indirect grouping strategy in one iteration. Small modifications of this procedure have been suggested by Goyal and Belton (1979) and Kaspi and Rosenblatt (1983). However, most heuristic solution procedures are iterative. Initially, the value of the basic cycle T is estimated, and then equations (2.3) and (2.2) are used, until the vector $(k_1, .., k_N)$ is unchanged in two successive iterations. Several authors, e.g. Goyal (1973, 1985, 1988), Kaspi (1991), and Kaspi and Rosenblatt (1985) suggested various ways to set the first value of T.

Kaspi and Rosenblatt (1985) compared the performance of several heuristics in an extensive simulation study. (Performance was measured by the average deviation from the cost of the optimal solution.) It turned out that a *combined approach,* that uses the non-iterative heuristic of Silver (1976) as starting point in the iterative algorithm of Goyal (1974b), gives the best results. The algorithm for the combined approach of Kaspi and Rosenblatt (1985) is outlined in Appendix 2.1.

We compare the computation time of the combined approach with that of the optimal method (Goyal (1974a)). For each item i, the parameters D_i, h_i, and a_i are randomly generated from uniform distributions over the range (100 , 100000), (0.2 , 2), and (1 , 51), respectively. Different values of A and N are considered. The computation times on a VAX-8700 computer for A=50 are tabulated in Table 2.1, together with the average percentage cost error of the heuristic. It turns out that the required computation time for the optimal solution method increases strongly as the number of items increases.

2.2.2 Direct grouping strategies

A direct grouping strategy is characterized by the following set of parameters:

M : number of groups;
S_j : set of items in group j (j=1,..,M);
T_j : time between two successive replenishments of all items in group j (j=1,..,M).

The direct grouping problem is to divide the set of N items in the family in M disjunct sets S_j with a replenishment cycle of T_j time units. (The number of groups, M, is predetermined in some studies. We consider the case where M is also a decision variable.) The average cost of a given direct grouping strategy with parameters $(M; S_1,..,S_M; T_1,..,T_M)$ is given by

$$TRC(M; S_1,..,S_M; T_1,..,T_M) = \sum_{j=1}^{M} \left\{ \frac{A + \sum_{i \in S_j} a_i}{T_j} + \frac{T_j}{2} \sum_{i \in S_j} D_i h_i \right\}. \tag{2.4}$$

The main difference between indirect grouping and direct grouping strategies is that the replenishment cycles of the groups formed by indirect grouping are multiple integers of some basic cycle time, whereas this is not the case for groups formed by direct grouping. It is implicitly assumed that no joint replenishments occur between items that are assigned to different groups. The effect of synchronizing the replenishments of groups is not considered. An exception to this approach is made by Chakravarty (1984) and Aggarwal (1984) who proposed heuristics to synchronize the replenishment cycles of groups, created by a direct grouping algorithm, such that the order cycle of any group is an integer multiple of the shortest order cycle. (Note that the resulting strategy belongs to the class of indirect grouping strategies.)

The optimal replenishment cycle for a given group j equals

$$T_j(S_j) = \sqrt{\frac{2(A + \sum_{i \in S_j} a_i)}{\sum_{i \in S_j} D_i h_i}}. \tag{2.5}$$

If (2.5) is substituted back in (2.4), we obtain

$$TRC(M; S_1,..,S_M) = \sum_{j=1}^{M} \sqrt{2 \cdot (A + \sum_{i \in S_j} a_i) \cdot \sum_{i \in S_j} D_i h_i} \; . \tag{2.6}$$

The problem of dividing N items into a variable number of groups is hard, because there are many combinations. Fortunately, it can be shown (see e.g. Bastian (1986)) that the optimal solution satisfies the so-called *consecutiveness property*: when the items are indexed in ascending (or descending) order of the ratio $a_i / D_i h_i$, then the optimal groups can be created from this sequential list. When two items from this list belong to the same group then all intermediate items also belong to this group.

Using the property of consecutive groups, dynamic programming recursion can be used to solve the direct grouping problem. Let f(k) denote the minimum cost to divide k items into a variable number of groups, and denote the number of items in the last group by z, then

$$f(k) = \min_{1 \le z \le k} \left(\sqrt{2 (A + \sum_{i=k-z+1}^{k} a_i) \sum_{i=k-z+1}^{k} D_i h_i} + f(k-z) \right) , \; f(0) = 0 \; , \tag{2.7}$$

where the items are chosen from the list of items consecutively ordered by $a_i / D_i h_i$.

However, the required computation time still increases strongly with the size of the problem. Page and Paul (1976), Goyal and Chakravarty (1984), Chakravarty and Goyal (1986), Chakravarty (1985), and Bastian (1986) proposed heuristic algorithms for direct grouping. However, except for the heuristic of Bastian, the joint ordering cost is not incorporated explicitly in the model. (Here, the problem is to divide the family of items in a predetermined small number of groups.)

Test examples show that the differences between Bastian's heuristic solution and the optimal solution from dynamic programming are very small. This simple heuristic starts with N consecutive groups (an individual item forms a group). Each iteration combines two neighbouring groups such that the decrease of the objective function is maximal. The procedure terminates as soon as the objective function can not be decreased by any combination of two neighbouring groups. A formal description of the algorithm of Bastian is given in Appendix 2.2.

We also consider the computation time needed for both the heuristic of Bastian and the optimal method, using the same problem settings as for the indirect grouping algorithms. The average computation time for different values of N, in case A=50, is shown in Table 2.1. As expected, the differences in computation time are considerable for large values of N. Table 2.1 shows that the computation times required for the heuristics for indirect grouping and direct grouping are comparable.

Table 2.1 Average performance over 1000 runs

N	Indirect grouping			Direct grouping		
	Computation time (milli-seconds)		Cost error (%)	Computation time (milli-seconds)		Cost error (%)
	optimal [1]	heuristic [2]		optimal [3]	heuristic [4]	
10	3.5	1.6	0.02	1.4	1.6	0.18
20	12.6	2.9	0.07	5.2	3.2	0.20
40	53.3	8.9	0.15	16.8	9.8	0.25
80	245.7	27.8	0.27	66.8	27.6	0.24

1=optimal method of Goyal (1974a);
2=combined approach of Kaspi and Rosenblatt (1985);
3=dynamic programming approach;
4=heuristic of Bastian (1986).

2.3 Experimental design and simulation results

Several inventory situations with constant demands are simulated to compare the performances of direct grouping and indirect grouping strategies. We analyze the differences between these two ways of grouping, and compare the performance of the strategies with the performance of an independent single-item strategy. Regression analysis is used to summarize the output of several simulation runs.

Kleijnen (1987) proposes the following hierarchical modelling approach: (a) determine the response or criterion variable of the study; (b) determine the independent variables or factors; (c) construct a regression metamodel (a cause-effect relation between the response variable and the independent variables of the simulation); (d) determine the experimental design (the situations that will be simulated); and (e) estimate the regression parameters and validate the metamodel; when the model is not valid step (b) or (c) is repeated; otherwise conclusions can be drawn.

Several authors have used simulation to study joint replenishment models (see e.g. Goyal and Satir (1989, p.11)). A popular response variable is the average cost saving of a joint replenishment strategy expressed as a percentage of the total cost of an independent EOQ strategy. This is a dimensionless variable, which is denoted by y_s. So if TRC_{eoq} denotes the total cost of all items under an independent EOQ strategy for each individual item, and TRC_s denotes the total cost of joint replenishment strategy s, then

$$y_s = 100 \cdot \frac{TRC_{eoq} - TRC_s}{TRC_{eoq}} \quad . \tag{2.8}$$

Relevant cost factors in the joint replenishment problem are the joint ordering cost (A), the individual ordering cost (a_i), and the inventory holding cost of stocking the periodic demand of item i for one period ($D_i h_i$). Other factors are: the number of items in the family (N) and the joint replenishment strategy s which is used.

Instead of blindly incorporating all these factors in a full fledged simulation experiment, these factors are first examined in pilot experiments.

Instead of all the individual numbers $D_i h_i$ and a_i, we can use the means $\overline{Dh} = \frac{1}{N}\sum_{i=1}^{N} D_i h_i$ and $\bar{a} = \frac{1}{N}\sum_{i=1}^{N} a_i$ in the simulation analysis. Pilot simulation experiments point out that other factors such as the variance of $D_i h_i$, and the variance of a_i are not important. Further, it appears that a different combination of the joint ordering cost (A) and the individual ordering cost (ā) with an equal ratio (A/ā) yields the same value of the response variable y_s. (This result can be proven formally for the special case that the joint ordering cost A and all individual ordering costs a_i increase with the same factor.) For this reason, we use the ratio (A/ā) instead of the joint ordering cost (A) and the average individual ordering cost (ā) separately. The value of the factor $\overline{Dh}$ does not affect the response variable y_s. (This result can also be proven formally for the special case that all numbers $D_i h_i$ increase with the same factor.) Therefore, the factor $\overline{Dh}$ is not a separate factor in the simulation.

Hence, it turns out that only two factors per replenishment strategy should be included in the metamodel: the ordering cost ratio (A/ā) and the number of items (N). A graphical analysis of the pilot experiments shows that an increase of the ordering cost ratio yields decreasing returns to scale and so does the number of items. Therefore we specify a regression metamodel with decreasing marginal percentage cost savings for the variables A/ā and N. Possible metamodels with decreasing marginal percentage cost savings are quadratic models, square root models, logarithmic models, and reciprocal models. All these models are linear in the parameters, so we can apply linear regression analysis to estimate the parameter vector of these regression models.

By definition, an experimental design determines which combinations of factor values are simulated. The choice of the experimental design is affected by the metamodel. Since in our case there are only two factors, a full factorial design can be used. The factor $A/\bar{a}$ is varied over six values; the factor N over four values; see Table 2.2. So, there are 24 different combinations. Every combination is simulated for both joint replenishment strategies, which gives $24 \cdot 2$ responses.

Table 2.2 Factors and levels

Factor	Levels					
$A/\bar{a}$	1	2	4	8	12	16
N	10	20	30	60		

Given a certain combination of $A/\bar{a}$ and N, the simulation program generates choices of the individual parameters: the joint ordering cost (A), and the individual values of a_i and $D_i h_i$. Individual values of a_i and $D_i h_i$ are randomly generated from a uniform distribution on the intervals (1 , 5) and (200 , 1800), respectively. The joint ordering cost is selected such that $A/\bar{a}$ is equal to the given value (thus, $A=3 \cdot A/\bar{a}$). Both direct grouping and indirect grouping are applied to the same inventory situation. Consequently, the responses (y_s) of different joint replenishment strategies s are based on the same random numbers. Each factor combination is replicated 500 times (a_i and $D_i h_i$ differ, whereas N and A are fixed). The performance of the strategies for the given factor combination is then measured by the cost savings (in %) averaged over 500 replications.

The simulation output of the 24 factor combinations is summarized by regression analysis. Since common random numbers have been used, the linear metamodels are estimated with Estimated Generalised Least Squares; see also Kleijnen (1987). We validate the models with Rao's lack of fit test (1959), Kleijnen's cross validation test (1988), and interpolation. We find that a logarithmic model fits and predicts the simulation data well within the range over which the two factors are varied. This yields equations (2.9) and (2.10), where standard deviations of the estimators are shown between parentheses. $\hat{y}_{dg}$ denotes the predicted cost saving (in %) of the direct grouping strategy and $\hat{y}_{ig}$ denotes the predicted cost saving (in %) of the indirect grouping strategy.

$$\hat{y}_{dg} = 6.6588 + \underset{(1.4E\text{-}05)}{15.9710} \ln (A/\bar{a}) + \underset{(2.3E\text{-}04)}{5.6209} \ln N \,, \tag{2.9}$$

$$\hat{y}_{ig} = 6.3064 + \underset{(1.6E\text{-}05)}{15.7797} \ln (A/\bar{a}) + \underset{(2.2E\text{-}04)}{5.9964} \ln N \,. \tag{2.10}$$

The interaction between the variables is not significant. We use Rao's F-test (1959) for linear hypotheses to see whether the effects of the independent variables are equal for both strategies. All coefficients differ significantly, because the standard deviations are virtually zero.

Figures 2.1 and 2.2 show the predicted responses $\hat{y}_{ig}$ and $\hat{y}_{dg}$ as a function of the ordering cost ratio and the number of items, respectively. Over the observed ranges of $A/\bar{a}$ and N the indirect grouping strategy always performs better than the direct grouping strategy does, but the difference is small. So the coefficients in equations (2.9) and (2.10) differ significantly but not importantly.

The estimate of the regression parameters shows that all coefficients in (2.9) are higher than in (2.10) except for the coefficient of N. So, the better performance of the indirect grouping strategy depends on the number of items in the family.

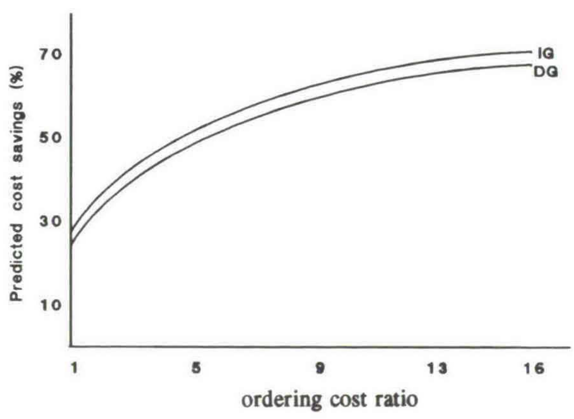

Figure 2.1 Predicted costs savings $\hat{y}_{dg}$ and $\hat{y}_{ig}$, as a function of the ordering cost ratio, given N=20. DG=direct grouping, IG=indirect grouping.

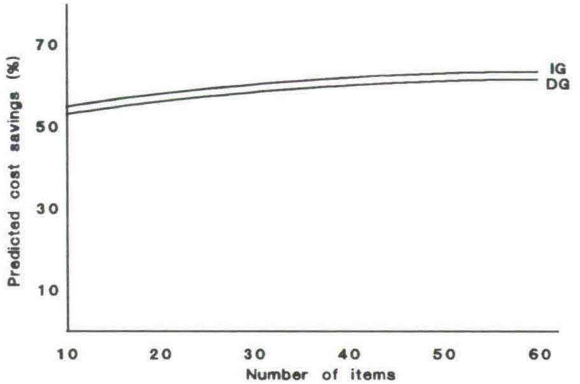

Figure 2.2 Predicted costs savings $\hat{y}_{dg}$ and $\hat{y}_{ig}$, as a function of the number of items in the family, given A/ā=20. DG=direct grouping, IG=indirect grouping.

It is not possible to extrapolate the logarithmic model to the left of the observed range, since for values of A/ā smaller than one, the variable ln (A/ā) will be negative. Extrapolation to the right of the observed range may result in responses $\hat{y}_i$ larger than hundred, which is impossible; see (2.8). So the metamodel is only valid for situations within the observed ranges.

Next, various situations are simulated with an ordering cost ratio larger than sixteen, the upper limit of the range in Table 2.2. Part a of Table 2.3 shows that the responses grow very slowly with an increasing ordering cost ratio when the ratio is higher than twenty-five. When the ratio is larger than seventy-five, the direct grouping and indirect grouping strategy become identical, because only one group is created.

We have already mentioned that indirect grouping strategies perform slightly better than direct grouping strategies within the observed range of Table 2.2. Table 2.3 (part b) shows that for very small values of the ordering cost ratio, direct grouping strategies perform better than indirect grouping strategies. With an ordering cost ratio of 0.01, the indirect grouping strategy performs even worse than the independent strategy does, because the replenishment cycles of the groups are restricted to an integer multiple of the basic cycle time. In this case the extra holding cost is greater than the joint ordering cost saved. In these situations, however, a joint replenishment strategy does not make much sense.

One of the purposes of our study is to find a threshold value of the joint ordering cost (relative to the average individual ordering cost) above which indirect grouping outperforms direct grouping. From Table 2.3 (part b) it follows that the threshold value of the ordering cost ratio is between 0.10 and 0.25 for N=20. The threshold value of $A/\bar{a}$ for different values of N is tabulated in part c of Table 2.3.

Table 2.3 Simulations with $A/\bar{a} > 16$ and $A/\bar{a} < 1$

Part a: Simulations with $A/\bar{a} > 16$ (N=20)			Part b: Simulations with $A/\bar{a} < 1$ (N=20)			Part c: Threshold value of $A/\bar{a}$	
$A/\bar{a}$	Cost savings (%)		$A/\bar{a}$	Cost savings (%)		Number of items	Threshold value
	DG	IG		DG	IG		
25	69.34	69.44	0.01	0.28	−0.56	10	0.08
50	72.73	72.74	0.05	1.78	1.33	20	0.14
75	73.94	73.94	0.10	3.66	3.50	30	0.20
100	74.59	74.59	0.25	8.87	9.24	40	0.30
500	76.25	76.25	0.50	15.76	16.56	50	0.42
1000	76.49	76.49	0.75	21.24	22.26	60	0.56

2.4 Conclusions

In this chapter we investigated two types of joint replenishment inventory strategies, namely indirect and direct grouping strategies, assuming constant demands. We presented a simulation design to study the effect of some factors that were expected to be important. The performances of the strategies were measured as the percentage cost saving of a joint replenishment strategy relative to an independent EOQ strategy. After some pilot experiments we concluded that only two factors are important, namely: (i) the ratio of the joint ordering cost and the average individual ordering cost, and (ii) the number of items in the family. Regression analysis was used to model the input-output behaviour of the simulation experiments with these two factors. A logarithmic model fitted and predicted the experimental data well within the range over which the two factors were varied. We performed also some extra simulation experiments outside the observed range.

The logarithmic metamodel showed that over the observed range of the experiments the indirect grouping strategy always outperforms the direct grouping strategy. The differences between the responses are, however, very small. The better performance of the indirect grouping strategy depends on the effect of the number of items in the family. The cost savings increase only slightly when the ordering cost ratio becomes larger than fifty. If the ratio is larger than seventy-five; only one group is created, and the direct grouping and indirect grouping strategy become identical. The threshold value of the ordering cost ratio under which direct grouping strategies outperform indirect grouping strategies is very small. If the ordering cost ratio is smaller than this threshold, then a joint replenishment strategy does not make much sense.

Appendix 2.1 Algorithm of the heuristic of Kaspi and Rosenblatt

As mentioned in Section 2.2, the heuristic of Kaspi and Rosenblatt (1985) is a combination of the non-iterative heuristic of Silver and the iterative heuristic of Goyal (1974b). Compared to formula (2.3), Goyal (1974b) proposed a slightly different approach to determine the optimal value of k_i $(i=1,..,N)$. His starting point is cost formula (A.2.1), which follows from substituting (2.2) into (2.1):

$$TRC(k_1,..,k_N) = \sqrt{2\left(A+\sum_{i=1}^{N}\frac{a_i}{k_i}\right)\left(\sum_{i=1}^{N}D_i h_i k_i\right)}\ . \tag{A.2.1}$$

The variable cost of item i is then minimized by selecting the integer k_i^* which satisfies

$$k_i^*(k_i^*-1) < \frac{a_i}{D_i h_i}\cdot\frac{\sum_{j\neq i}D_j h_j k_j}{A+\sum_{j\neq i}\frac{a_j}{k_j}} \leq k_i^*(k_i^*+1)\ . \tag{A.2.2}$$

Note that condition (A.2.2) is the same as (2.3) if

$$T = T_{(-i)}(k_1,..,k_N) = \sqrt{\frac{2\left(A+\sum_{j\neq i}\frac{a_j}{k_j}\right)}{\sum_{j\neq i}D_j h_j k_j}}\ ,\quad i=1,..,N\ . \tag{A.2.3}$$

Algorithm for indirect grouping

Step 1: Determine $r := \arg\min_i\ a_i/D_i h_i$. Set $k_r := 1$.
Calculate

$$T_0 := \sqrt{\frac{2(A+a_r)}{D_r h_r}}\ .$$

Determine $k_i = L$ which satisfies $L(L-1) < \dfrac{2a_i}{D_i h_i T_0^2} \leq L(L+1)$,

for $i=1,..,r-1,r+1,..,N$.

Step 2: Calculate $T_{(-i)}$ with (A.2.3) for $i=1,..,N$.

Step 3: Determine $k_i^*(T_{(-i)})$ with (2.3) for $i=1,..,N$.

Step 4: Go to Step 5 if the vector $(k_1,..,k_N)$ is unchanged in two successive iterations; otherwise return to Step 2.

Step 5: Calculate the basic cycle time with (2.2).

Appendix 2.2 Algorithm of the heuristic of Bastian

The heuristic of Bastian (1986) has already been discussed in Section 2.2. The heuristic combines in every iteration two neighbouring groups such that the decrease of the objective function is maximal. The procedure terminates as soon as the objective function can not be decreased by any combination of two neighbouring groups. Denote by μ_j the increase of the objective function when group j and j+1 are combined. It is easily seen that

$$\mu_j = \sqrt{2\Big(A + \sum_{i \in S_j \cup S_{j+1}} a_i\Big) \cdot \sum_{i \in S_j \cup S_{j+1}} D_i h_i} - \sqrt{2\Big(A + \sum_{i \in S_j} a_i\Big) \cdot \sum_{i \in S_j} D_i h_i} - \sqrt{2\Big(A + \sum_{i \in S_{j+1}} a_i\Big) \cdot \sum_{i \in S_{j+1}} D_i h_i} \, . \tag{A.2.4}$$

Algorithm for direct grouping

Step 1: Index the items in descending order of their ratio $D_i h_i / a_i$ $(i=1,..,N)$.

Step 2: Set $S_j := \{j\}$ and $M := N$.
Calculate μ_j for $j=1,..,N-1$ with (A.2.4).

Step 3: Go to Step 6 if $\min_j \mu_j > 0$; else go to Step 4.

Step 4: $M := M-1$.
Determine $k := \arg\min_j \mu_j$.
Combine group k en k+1: $S_k := S_k + S_{k+1}$.
Put the groups in order.
Go to Step 6 if $M=1$; otherwise go to Step 5.

Step 5: Calculate μ_{k-1} and μ_k (if they exist).
Return to Step 3.

Step 6: Determine T_j with (2.5) for $j=1,..,M$.

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CHAPTER 3

ON THE DETERMINATION OF THE OPTIMAL INDIRECT GROUPING STRATEGY

This chapter considers the joint replenishment problem under constant demand. We investigate the cyclic indirect grouping strategy: a family replenishment is made every T time units and item i is included in each k_ith replenishment. Goyal proposed a solution to find the global optimum within this class of cyclic strategies. However, it will be shown that the algorithm of Goyal does not always lead to the optimal cyclic strategy. A simple correction is suggested.

3.1 Introduction

In Chapter 2 we considered a multi-item inventory problem with a so-called joint ordering cost structure: a joint ordering cost is incurred for each replenishment, independent of which items are involved. In addition, an individual ordering cost is incurred for each item included in the replenishment. So, cost savings can be achieved by coordinating the replenishments of several items. Two classes of strategies for the constant demand case have been compared: indirect and direct grouping strategies.

In this chapter we will investigate the determination of the parameters of the optimal indirect grouping strategy. Recall that under this cyclic strategy an order is placed every T time units and item i is included in the family order every k_ith replenishment. The ordering strategy leads to a replenishment interval of $T \cdot k_i$ time units for the *i*th item. Note that (actual) joint replenishments are equally spaced under such a strategy only if $k_{min}=1$, where k_{min} is the minimum value of k_i over all items i. The cyclic strategies $(T;k_1,..,k_N)$ with $k_{min}=1$ are called *strict cyclic*.

This chapter is a modified version of "A note on the joint replenishment problem under constant demand", which has been published in *Journal of the Operational Research Society* 44, 1993, 185-191.

Example 3.1

Let N=2, k_1 =2, k_2 =3, T=1. Table 3.1 shows the items which are included in the order on subsequent time intervals. Note that no order is placed on t=1 and t=5. The actual joint replenishments are not equally spaced.

Table 3.1 Items which are included in the order at specific replenishment dates

time	0	1	2	3	4	5	6
order	{1,2}	{ }	{1}	{2}	{1}	{ }	{1,2}

Andres and Emmons (1976) have shown by a counter example that the overall optimal strategy for the joint replenishment problem is not necessarily a strict cyclic strategy. They illustrate this statement with a two product example, for which the optimal solution is obtained by a special algorithm (see Andres and Emmons (1975)). The problem setting of Andres and Emmons (1975) is different from that of the joint replenishment problem in the following sense: a joint ordering cost has to be paid at every replenishment in which not all items are involved. The only way to avoid the joint ordering cost is to perform a joint replenishment for all items simultaneously (the problem settings are only equivalent if the number of items is two).

Goyal (1974) has proposed an algorithm to find the optimal solution within the class of cyclic strategies. It will be shown in Section 3.2 that the solution, which is found by this algorithm is not necessarily optimal within the class of cyclic strategies. In Section 3.3 we propose a modification of Goyal's algorithm to enable cyclic strategies with $k_{min} > 1$. Some computational aspects of the modified algorithm are discussed in Section 3.4. Section 3.5 summarizes the conclusions.

3.2 Cyclic (indirect grouping) strategies

The joint replenishment problem with a joint ordering cost structure has been described explicitly in Section 2.2. (The notation is also introduced in this section.) Recall that the total average cost per time unit for an arbitrary cyclic strategy $(T;k_1,..,k_N)$ is given by

$$TRC(T;k_1,..,k_N) = \frac{1}{T}(A + \sum_{i=1}^{N} \frac{a_i}{k_i}) + \frac{T}{2}\sum_{i=1}^{N} D_i h_i k_i \ . \tag{3.1}$$

We assume that for a cyclic strategy with $k_{min} > 1$ the joint ordering cost is also incurred at multiples of T at which no actual replenishment is performed. Dagpunar (1982) reformulated the objective function of the (T; k_1 ,..,k_N) strategies under the assumption that no joint ordering cost is charged if no order is placed. However, as pointed out in Goyal (1982) the minimization of the cost function, provided by Dagpunar, is considerably more complex than that of the original problem.

It is easy to obtain the optimal value $T^*(k_1, .., k_N)$ of the basic cycle time for a given parameter vector (k_1 ,..,k_N); see (2.2). If $T^*(k_1, .., k_N)$ is substituted back in the original cost function (3.1) the following cost function is obtained:

$$TRC(k_1,..,k_N) = \sqrt{2(A + \sum_{i=1}^{N} \frac{a_i}{k_i})(\sum_{i=1}^{N} D_i h_i k_i)} \, . \tag{3.2}$$

Most solution procedures solve a subproblem in which the values of k_i are not necessarily restricted to integer variables. Schweitzer and Silver (1983) have shown for this continuous variable case that the problem is ill-posed. (They show that the infimum of the objective function occurs at a boundary point (k_i=0, i=1,..,N) that is not feasible.) When the restriction $k_i \geq 1$ is imposed, it can easily be shown (see Appendix 3.1) that $k_{min}=1$ in the optimal solution of the continuous variable case. However, the variables k_i are not continuous but integer in the original problem. For the mixed-integer case the following result holds:

The optimal solution within the class of cyclic strategies does not necessarily belong to the class of strict cyclic strategies.

The proof is provided by a counter example.

Example 3.2

Let N=2, A=1, D_1=400, D_2=900, a_1=50, a_2=50, h_1=1, h_2=1 (the data are taken from Andres and Emmons (1976)). The optimal strict cyclic strategy, obtained by Goyal's algorithm is $k_1^*=2$, $k_2^*=1$, and $T^*=0.30$. The cost of this strict cyclic strategy is 508.33. However, the optimal cyclic ($T^*;k_1^*,..,k_N^*$) strategy is $k_1^*=3$, $k_2^*=2$ and $T^*=0.17$, which is not strict cyclic. The corresponding cost is 505.96.

Note that if $k_{min} > 1$ and the strategy is cyclic, then it is profitable to order nothing on some specific replenishment dates, whereas the joint ordering cost still has to be paid.

3.3 Modification of the algorithm of Goyal

In the literature regarding to the joint replenishment procedures for cyclic strategies, the algorithm developed by Goyal (1974) is used to find the optimal cyclic strategy. However, the algorithm only guarantees an optimal strict cyclic strategy. To make this clear we shortly review Goyal's original algorithm.

Goyal has derived that for a fixed value of T the variable costs of item i are minimised by selecting the integer $k_i^*(T)$ which satisfies (2.3). Hence, in the optimal solution the following set of conditions has to be satisfied:

$$\frac{2a_i/D_ih_i}{k_i(T)(k_i(T)+1)} \leq T^2 < \frac{2a_i/D_ih_i}{k_i(T)(k_i(T)-1)}\ ,\ i=1,..,N\ . \tag{3.3}$$

Goyal's algorithm first determines a minimum (T_{min}) and a maximum value (T_{max}) for T and then, using (3.3), it determines all intervals of T within this range for which the vector $(k_1,..,k_N)$ is unchanged. It can be shown that only a finite number of intervals has to be considered. Hence the global optimum can be obtained by taking the minimum of all local minima after explicit enumeration of all the intervals.

Since T $(k_1,..,k_N)$ is monotone decreasing in each component k_i the maximum value of $T^*(k_1,..,k_N)$ occurs in $(k_1,..,k_N)=(1,..,1)$. Goyal (1974) stated that the minimum of T is equal to the minimum of $(2a_i/D_i h_i)^{½}$ over all i. Andres and Emmons (1976) already noted that this lower bound on T, however, is not correct. They showed that a correct lower bound T_{min} is given by $\min_i\ (a_i/D_i h_i)^{½}$, provided that attention is restricted to strict cyclic strategies. (In Goyal (1988), an example is given where the original algorithm does not give the optimal solution. However, if T_{min} is set equal to the minimum of $(a_i/D_i h_i)^{½}$ over all i, then the optimal solution is also found with the original algorithm of Goyal (1974).)

Note that T_{min} follows from formula (3.3) with $k_i=1$. So, T_{min} is based on the assumption that at least one of the items has an optimal k_i value of one. Our criticism on Goyal's algorithm is that this choice of T_{min} excludes a cyclic $(T;k_1,..,k_N)$ strategy with $k_{min} > 1$. Such a strategy is not strict cyclic. Since the optimal cyclic strategy does not necessarily belong to the class of strict cyclic strategies, the strategy obtained from Goyal's algorithm may not be the optimal cyclic strategy.

To handle this problem, we suggest to use another lower bound T_{min}. Denote the cost and the basic cycle time of the optimal strategy by TRC^* and T^* respectively and recall that the joint ordering cost is denoted by A. From formula (2.2) and (3.2) it follows that $T^* > 2A/TRC^*$. The problem is that TRC^* is unknown. Let TRC be the cost of an arbitrary feasible $(T;k_1,..,k_N)$ strategy, then it is obvious that $TRC \geq TRC^*$. This implies:

$$T^* > \frac{2A}{TRC} . \tag{3.4}$$

So 2A/TRC provides a lower bound for T^* for any feasible strategy with parameters $(T;k_1,..,k_N)$. The following modified version of Goyal's algorithm, which uses this lower bound, will always find the optimal cyclic strategy.

First, define:

$(k_1^{\sim},..,k_N^{\sim})$: a vector which keeps up with the best relative replenishment frequencies, which are found so far;

$TRC^{\sim}$: the cost corresponding to $(k_1^{\sim},..,k_N^{\sim})$.

Modified algorithm of Goyal

Step 1: Initialisation:

a) Set $(k_1^{\sim},..,k_N^{\sim}) := (1,..,1)$.
Determine $TRC^{\sim} := TRC(1,..,1)$ with (3.2).

b) Set $T_{max} := TRC^{\sim} / \Sigma_{i=1}^{N} D_i h_i$ (this follows from $TRC^* / T^* = \Sigma_{i=1}^{N} D_i h_i k_i^*$).
Determine $(k_1,..,k_N) := (k_1(T_{max}),..,k_N(T_{max}))$ with (2.3).
Determine $TRC(k_1,..,k_N)$ with (3.2).

c) If $TRC(k_1,..,k_N) < TRC^{\sim}$, then
- $(k_1^{\sim},..,k_N^{\sim}) := (k_1,..,k_N)$;
- $TRC^{\sim} := TRC(k_1,..,k_N)$;
- return to Step 1b;

otherwise go to Step 1d.

d) Set $T_{min} := 2A/TRC^{\sim}$.
Set $T_{ch}(i) := (2a_i/(D_i h_i \cdot k_i(k_i+1)))^{1/2}$ for all items i ($T_{ch}(i)$ denotes the value of the basic cycle at which the value of k_i of item i changes to k_i+1).

Step 2: Set $T := \max T_{ch}(i)$.
If $T \leq T_{min}$, then go to Step 4; otherwise go to Step 3.

Step 3: Evaluation of the cost in the new interval:
Set $p := \arg_i \max T_{ch}(i)$ (p is the item for which k_i changes to $k_i + 1$ on time T).
Set $k_p := k_p + 1$, and set $T_{ch}(p) := (2a_p/(D_p h_p \cdot k_p(k_p+1)))^{1/2}$.
Calculate $TRC(k_1,..,k_N)$ with (3.2).
If $TRC(k_1,..,k_N) < TRC^{\sim}$, then
- $(k_1^{\sim},..,k_N^{\sim}) := (k_1,..,k_N)$;
- $TRC^{\sim} := TRC(k_1,..,k_N)$;
- $T_{min} := 2A/TRC^{\sim}$.

Return to Step 2.

Step 4: Termination of the algorithm:
The optimal cyclic strategy has parameters $(k_1^*,..,k_N^*) := (k_1^{\sim},..,k_N^{\sim})$ and T^*, where T^* follows from (2.2).

Example 3.2 (continued)
Recall that Goyal's original algorithm yields $k_1^* = 2$, $k_2^* = 1$, and $T^* = 0.30$ with cost 508.33. The adjusted algorithm yields $k_1^* = 3$, $k_2^* = 2$, and $T^* = 0.17$. The corresponding cost is 505.96. This solution is the optimal cyclic $(T^*; k_1^*,..,k_N^*)$ strategy. The overall optimal strategy for the joint replenishment problem is to order item 1 two times per time unit and item 2 three times per time unit (see Andres and Emmons (1976)). Note that this solution corresponds with a cyclic strategy with $T=0.17$, $k_1 = 3$, and $k_2 = 2$. The cost of this strategy, 504, is lower than the cost obtained by the modified algorithm, because it is assumed that every T time units the joint ordering cost is charged regardless whether an actual order is placed. In the model of Andres and Emmons the joint ordering cost is only charged when any of the items of the family is actually ordered.

3.4 Computational aspects

A simulation program has been developed to compare the solution and the computation time of the original and the modified version of Goyal's algorithm. (In the experiments the minimum value of $(a_i/D_i h_i)^{1/2}$ over all items i is used as a lower bound on the basic cycle in the original algorithm.) The demand (D_i), the holding cost (h_i), and the individual ordering cost (a_i) for each item are randomly generated from a uniform distribution with range (100, 100000), (0.2, 2), and (1, 51), respectively. Three different levels of the number of items (N= 5, 10, 20) and four different levels of the joint ordering cost (A= 1, 5, 10, 50) are considered. For each combination of N and A, 1000 examples are generated and solved by the algorithms. The third column of Table 3.2 presents the number of examples, where the optimal strategy does not belong to the class of strict cyclic strategies. The fourth and the fifth column show the average error and the maximum error of the cost obtained by the original algorithm of Goyal relative to the cost of the optimal cyclic solution (in case the optimal cyclic strategy is not strict cyclic).

It appears that the original algorithm of Goyal does not provide the optimal cyclic strategy in many cases, if the joint ordering cost (A) is low relative to the average individual ordering cost ($\bar{a}$=25). Table 3.2, however, shows that the average error and the maximum error are rather small. The optimal cyclic strategy was strict cyclic in all our examples if the ordering cost ratio (A/$\bar{a}$) is larger than 0.2. This can be explained, because in these situations it will be too costly to order nothing on a particular multiple of T. Recall that the class of non-cyclic direct grouping strategies (with unequal spaced family replenishments) outperforms the class of cyclic indirect grouping strategies for small ordering cost ratios (see Chapter 2).

Table 3.2 Summary results of 1000 examples

N	A	number of examples, where the optimal strategy is not strict cyclic	average error (%)	maximum error (%)
5	1	193	0.30	0.91
5	5	0	0.00	0.00
5	10	0	0.00	0.00
5	50	0	0.00	0.00
10	1	320	0.33	1.12
10	5	7	0.13	0.38
10	10	0	0.00	0.00
10	50	0	0.00	0.00
20	1	420	0.22	1.03
20	5	36	0.18	0.51
20	10	0	0.00	0.00
20	50	0	0.00	0.00

Table 3.3 presents the average computation time on a VAX-8700 computer for a given combination of N and A. Note that the adjusted value of T_{min} in the modified

algorithm depends heavily on the cost input-data. It is clear that the same holds for the computation time, which is needed to find the optimal cyclic strategy. In our examples, the computation time of the modified algorithm increases dramatically compared with the computation time of the original algorithm if the joint ordering cost is small.

Table 3.3 Average computation time over 1000 examples (expressed in milli-seconds)

N	A	original algorithm of Goyal	modified algorithm of Goyal
5	1	0.97	43.26
5	5	1.04	8.92
5	10	0.94	4.74
5	50	1.09	1.36
10	1	3.51	244.52
10	5	2.97	48.91
10	10	2.88	24.54
10	50	3.04	5.78
20	1	13.01	1462.83
20	5	12.56	288.08
20	10	12.98	151.53
20	50	12.26	30.08

Next we conclude that the difference in computation time, needed by the modified algorithm and the original algorithm decreases if the joint ordering cost increases (note that $T_{min}:=2A/TRC$). Some extra simulation runs (see Table 3.4) show that the computation time needed by the modified algorithm is even smaller than the computation time of the original algorithm if the joint ordering cost is very large.

Table 3.4 Average computation time over 1000 examples (expressed in milli-seconds)

N	A	original algorithm of Goyal	modified algorithm of Goyal
5	500	0.95	0.49
10	500	3.44	0.93
20	500	12.61	3.16

Based on this observation we recommend to use the following dynamic lower bound on T if the objective is to find the optimal *strict* cyclic strategy:

$$T_{\min} = max \left\{ min_i \sqrt{\frac{a_i}{D_i h_i}} , \frac{2A}{TRC^{\sim}} \right\} , \tag{3.5}$$

where TRC $^{\sim}$ denotes the cost of the best strategy which is found so far.
Thus, if the modified algorithm is used, where $T_{\min}$ is obtained by formula (3.5), then the computation time, needed to find the best strict cyclic strategy, may be smaller compared with the computation time of the original algorithm.

3.5 Conclusions

We considered the class of cyclic $(T;k_1 ,..,k_N)$ strategies for the joint replenishment problem. The smallest k_i value is denoted by $k_{\min}$. The class of cyclic strategies with $k_{\min}=1$ is called strict cyclic. Under a strict cyclic strategy the actual family replenishments and the replenishments of the individual items in the family are equally spaced (by respectively T and $T \cdot k_i$ time units). Under a cyclic strategy with $k_{\min} > 1$ fake replenishments are made at some multiples of T. The joint ordering cost is incurred at all multiples of T, even if there is no actual replenishment. In the literature it is assumed that the optimal cyclic strategy belongs to the class of strict cyclic strategies ($k_{\min}=1$). We have illustrated that the optimal cyclic strategy is not necessarily strict cyclic. Goyal's algorithm, which is commonly used to find the optimal cyclic strategy, does not allow solutions with $k_{\min} > 1$. As a consequence, this algorithm yields the optimal strict cyclic strategy, but it does not always yield the best cyclic strategy. We proposed a simple modification of the lower bound in Goyal's algorithm. The modified algorithm, which uses a dynamic lower bound, does always find the optimal cyclic $(T;k_1 ,..,k_N)$ strategy. Simulation results showed that the optimal cyclic strategy is not strict cyclic only if the ratio of the joint ordering cost and the average individual ordering cost is rather small. It turned out that the computation time needed depends heavily on the cost input data. In the examples considered the computation time of the modified algorithm increased dramatically compared with that of the original algorithm for small values of the joint ordering cost. On the other hand, the modified algorithm needed less computation time for large values of the joint ordering cost.

Appendix 3.1 The continuous variable case

Lemma 3.1

$k_{min}=1$ in the optimal solution of the continuous case.

Proof

Let strategy 1 be of the form: $k_i^{(1)}>1$ for all items i, and let $T^{(1)}$ be the corresponding basic cycle time. The cost is denoted by $C^{(1)}$.

Consider a new strategy 2 with $k_i^{(2)}=k_i^{(1)}/k_{min}^{(1)}$, $i=1,..,N$, and $T^{(2)}=T^{(1)}\cdot k_{min}^{(1)}$. Note that for this second strategy $k_{min}^{(2)}=1$. It follows from formula (3.1) that $C^{(2)}$, the cost of strategy 2, equals:

$$C^{(2)} = \frac{1}{T^{(1)}k_{min}^{(1)}}\left(A+\sum_{i=1}^{N}\frac{a_i}{k_i^{(1)}/k_{min}^{(1)}}\right)+\frac{T^{(1)}k_{min}^{(1)}}{2}\sum_{i=1}^{N}D_i h_i\frac{k_i^{(1)}}{k_{min}^{(1)}} =$$
$$\frac{A}{T^{(1)}k_{min}^{(1)}}+\frac{1}{T^{(1)}}\sum_{i=1}^{N}\frac{a_i}{k_i^{(1)}}+\frac{T^{(1)}}{2}\sum_{i=1}^{N}D_i h_i k_i^{(1)} . \tag{A.3.1}$$

Now, we can conclude that $C^{(2)}<C^{(1)}$, since $k_{min}^{(1)}>1$.

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CHAPTER 4

CAN-ORDER POLICIES WITH DISCOUNT EVALUATION

In many practical situations, coordination of replenishment orders for a family of items can lead to considerable cost savings. A well-known class of strategies for the case where cost savings are due to reduced joint ordering costs is the class of can-order strategies. However, these strategies, which are easy to implement in practice, do not take discount possibilities into account. We propose a method to incorporate discounts in the framework of can-order strategies. A continuous review multi-item inventory system is considered with independent compound Poisson demand processes for each of the individual items. Discounts are offered by the supplier as a percentage of the total dollar value whenever this value exceeds a given threshold. Starting from the can-order strategy as a basic decision rule, we develop a simple heuristic to evaluate these discount opportunities.

4.1 Introduction

The main part of inventory management literature is devoted to single-item models. These models neglect possible savings which can be achieved by ordering two or more items together. These savings can be caused by reduced ordering costs, reduced freight rates, quantity discounts or improvement of the implementation of stock control. Therefore, joint replenishment models, in which the interaction among different items is explicitly taken into account, are very useful in many practical situations.

So far, the literature on joint replenishment models has been almost exclusively confined to settings where economies of scale in joint replenishments are restricted to ordering costs only. In this situation, a realistic way to model the cost effectiveness of coordination is by the joint ordering cost structure, where a joint ordering cost is incurred for any order and an individual ordering cost is incurred for each item in the replenish-

This chapter is based on a part of the paper "Coordinated replenishment systems with discount opportunities", co-authored by F.A. van der Duyn Schouten and R.M.J. Heuts, which has been submitted to *International Journal of Production Research*.

ment. The joint ordering cost can be considered either as the fixed cost of placing an order, independent of the number of items in the order, or a fixed transportation cost per ride. The individual ordering cost can be considered as the relatively minor cost of adding one more item to the order. The constant demand case for this kind of joint replenishment problem has been discussed in Chapters 2 and 3. For the case of stochastic demand processes, an overview of inventory control policies has been given in Chapter 1.

Other reasons for coordinated control include quantity or freight rate discounts. Freight rate discount schedules are quite similar to quantity discount schedules, such as all-units or incremental-units discount schedules, but are usually based on weight, volume, car loads, or standard container sizes, instead of units or dollars. In many cases it may be uneconomical to achieve a discount breakpoint quantity by ordering one single item, but it could certainly make sense to coordinate several items to achieve such a discount breakpoint. In a sequence of papers, Miltenburg and Silver (1984a, 1984b, 1984c, 1988) and Miltenburg (1985, 1987) proposed a control strategy for multi-item inventory systems with discount opportunities. They have shown that their system outperforms the more widely known IMPACT system of IBM (we refer to IBM (1971) and Kleijnen and Rens (1978) for more details on the latter system).

In this chapter an alternative method is proposed that accounts both for joint ordering costs and quantity discounts. In Section 4.2 the model is described in detail and the notation is introduced. Starting from a basic ordering strategy we describe a class of simple discount evaluation procedures in Section 4.3. In Section 4.4 an algorithm is presented to find the best policy within this class. Some numerical examples are presented in Section 4.5 to validate the optimization procedures. Finally, the conclusions are listed in Section 4.6.

4.2 Description of the problem

Coordinated replenishment systems for a family of items with continuous review are considered, where demands are generated by independent compound Poisson processes for each item. Demand sizes for item i are independent random variables with a given probability distribution. Excess demands are backordered and the lead time is constant (the generalisation to stochastic lead times is straightforward).

The objective is to minimize the total expected long run average cost per unit of time subject to a given service level constraint. The relevant costs consist of ordering, holding, and purchasing costs. The ordering costs are divided into two parts: a joint ordering cost is charged whenever an order is triggered. An individual ordering cost is added if item i is included in the order. Holding costs are charged at a constant rate per unit of time on every unit of item i on stock. When the total (dollar) value of a replenishment is greater than or equal to a given threshold value, a percentage discount of the total value of the replenishment is awarded. (This discount schedule is referred to as the all-units discount schedule in the literature.)

The following notation will be used:

N : number of items in the family;
λ_i : Poisson arrival rate of customers for item i $(i=1,..,N)$;
m_i : maximum demand size for item i $(i=1,..,N)$;
$\phi_i(j)$: probability that a demand for item i equals j units $(i=1,..,N, j=1,..,m_i)$;
L : lead time;
A : joint ordering cost per replenishment (independent of the items in the replenishment or the number of units ordered);
a_i : individual ordering cost for item i when it is included in the replenishment (independent of the order quantity) $(i=1,..,N)$;
h_i : holding cost of item i per unit per unit time $(i=1,..,N)$;
w_i : unit purchasing cost of item i $(i=1,..,N)$;
I_i : inventory position of item i $(i=1,..,N)$;
Q : total value of the family replenishment (in dollars);
Q_d : discount threshold (in dollars);
d : discount percentage $(0 \leq d \leq 1)$.

4.3 Can-order policies with discount opportunities

For the problem described in the previous section, it is usually impossible to find the overall optimal policy. Moreover, if it can be obtained, it will result in complex decision rules which are hard to implement in practical situations. Therefore, in the literature a lot of attention is paid to nearly optimal strategies, which are more easy to implement. For the case that savings by joint replenishments are caused by reduction of ordering costs only, the class of *can-order policies* has been proposed. Under this inventory control rule, which is characterized by three parameters (S_i, c_i, s_i) for each item i, the inventory position is continuously reviewed. Whenever the inventory position of any item i drops to or below its must-order point (s_i), a family replenishment is made. All the other items j with an inventory position less than or equal to their can-order point (c_j) are included in this replenishment. The inventory position of each item k in the replenishment is then raised to the order-up-to level (S_k).

Silver (1974, 1981), Thompstone and Silver (1975), and Federgruen et al. (1984) have proposed heuristic solution procedures to find the parameters of the optimal can-order strategy for the case of (compound) Poisson demands. The determination of the parameters is based on a trade-off among the ordering cost, the holding cost, and the required service level. Under the absence of discounts the purchasing cost of each item is irrelevant, since the sum of these costs per unit time is constant under all reasonable policies. However, when discount opportunities occur the purchasing costs have to be taken into account. For this situation we propose the following heuristic approach:

CAN$^+$ system: can-order policy with discount evaluation

Step 1: Ignoring quantity discounts compute the optimal can-order policy with the method of Federgruen et al. (1984).

Step 2: At each demand epoch at which an order is triggered according to the can-order policy from Step 1, the composition of the order is determined via a one-stage optimization procedure, which incorporates the potential for exploiting the quantity discount.

Before implementing Step 2 of the heuristic, we have to describe in more detail the procedure of Federgruen et al. (1984) as used in Step 1. The inventory position of a particular item i under the control mechanism of the can-order strategy is shown in Figure 4.1. Note that an individual item i is ordered whenever $I_i \leq s_i$ (at T_1) or when $I_i \leq c_i$ while $I_j \leq s_j$ for any other item $j \neq i$ (at time T_3). When an item $j \neq i$ triggers a replenishment, we call this a *special replenishment opportunity* for item i. (So, in Figure 4.1, special replenishment opportunities occur at time T_2 and T_3.)

Although the control mechanism of the can-order strategies is very simple, it is difficult to determine the optimal control parameters (S_i, c_i, s_i) for $i=1,..,N$. Interaction is caused by the special replenishment opportunities. The basic idea behind the heuristics for this problem is to reduce the multi-item system to a number of appropriately chosen single-item problems. Crucial is the assumption that the special replenishment opportunities for item i (the trigger moments of all other items) can be approximated by a Poisson process. The rate of this process, μ_i, is equal to sum of the expected number of

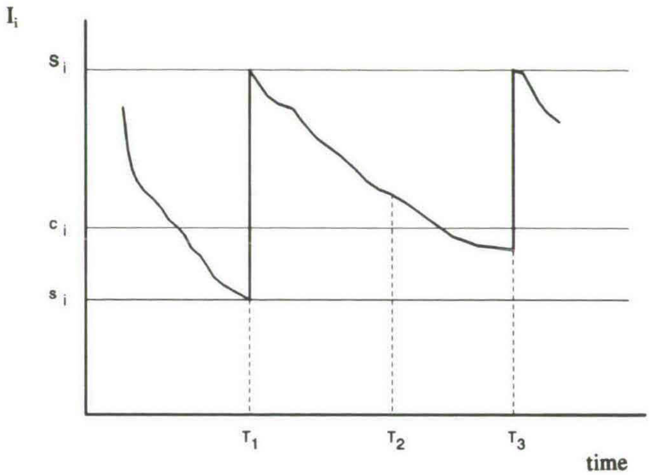

Figure 4.1 Inventory position of an item under a can-order strategy

trigger events per unit time of the other items. Let ξ_j denote the expected number of replenishments per unit time triggered by item j, then $\mu_i := \Sigma_{j \neq i} \xi_j$. The single-item problem for item i has *normal* replenishment opportunities, with ordering cost equal to $A + a_i$, at demand epochs for the item; in addition, there are *special* replenishment opportunities, with reduced ordering cost a_i, at epochs which are generated by a Poisson process with rate μ_i. Hence, given a set of trigger intensities (ξ_j), the control parameters (S_i, c_i, s_i) are determined from the solution of the single-item problem for item i. An iterative solution procedure is then used since the control parameters of item i influence the rate of special replenishment opportunities of any other item. The iteration process stops when the control parameters are the same in two subsequent iterations. (For a more detailed description of this approach, we refer to Chapter 6. In particular, the simplifying assumption of the Poisson arrival process for special replenishment opportunities is discussed in that chapter.)

Federgruen et al. (1984) solve the single-item model for item i by defining an appropriate semi-Markov decision model, with decision epochs the moments at which either a demand or a special replenishment opportunity for item i occurs. The state of the system of this single item model is represented by (x,z), the inventory position x of item i just after a demand ($z=0$) or after a special replenishment opportunity ($z=1$). A special policy-iteration algorithm is used to find the optimal control parameters for the particular item.

From semi-Markov decision theory, we know that the policy-iteration procedure not only yields the optimal policy, but also the so-called *relative values* (see e.g. Tijms (1986)), denoted by $v_i(x,z)$. In particular the difference between the relative values $v_i(x,z)$ and $v_i(x',z')$ denotes the difference in total expected costs over an infinitely long period when starting in state (x,z) as compared to starting in state (x',z'). We will use these relative values as an approximation of the future cost differences after the first demand arrival, due to different ordering decisions at the current replenishment epoch.

4.4 A computational procedure for discount evaluation

In Step 2 of our heuristic a decision has to be made about the order composition. First, initial order sizes are obtained by the basic can-order strategy. If the value of the initial order sizes is enough to achieve the discount, then the order composition is determined by this regular can-order replenishment. However, if the dollar value is not enough, then an additional decision has to be made whether the regular can-order replenishment has to be enlarged to achieve the discount threshold. Note that such an enlargement affects not only the (direct) costs at the current epoch, but also future decisions (and hence future costs). The order composition is therefore based on a comparison of direct and future costs of all possible ordering decisions. The direct costs include the ordering and purchasing cost of the ordering decision as well as the inventory holding costs until the first subsequent demand epoch, which is a demand for item i with probability λ_i/Λ, where $\Lambda=\Sigma_j \lambda_j$. As approximation for the future costs, including all costs after the first subsequent demand epoch, we use the relative values $v_i(x,z)$. The only difference with the relative values as obtained by Federgruen et al. (1984) is that in the situation with discount opportunities the purchasing costs have to be taken into account in the cost function explicitly. The details of the computation of the relative values $v_i(x,z)$ are presented in Appendix 4.1.

Given the actual inventory position I_i of all items i and given a fixed can-order strategy (S_i, c_i, s_i), the following non-linear knapsack problem (**KS**) has to be solved to determine the optimal order composition *when the discount is taken*. (The order quantity of item i is denoted by q_i.)

Problem KS

$$\min_{(q_1,..,q_N)} \sum_{i=1}^{N} \{ \delta(q_i)\, a_i + (1-d)\, q_i w_i + \frac{h_i}{\Lambda} \sum_{k=0}^{I_i+q_i} (I_i+q_i-k)\, r_i(k) + \tag{4.1}$$
$$\frac{\lambda_i}{\Lambda} \sum_{j=1}^{m_i} \phi_i(j)\, v_i(I_i+q_i-j,0) + (1-\frac{\lambda_i}{\Lambda})\, v_i(I_i+q_i,0) \},$$

s.t.

$$q_i + I_i \geq S_i \quad , \quad if\ I_i \leq c_i \quad , \quad i=1,..,N, \tag{4.2}$$

$$q_i + I_i \leq UB_i \quad , \quad i=1,..,N, \tag{4.3}$$

$$\sum_{i=1}^{N} q_i w_i \geq Q_d \ , \tag{4.4}$$

$$q_i \geq 0 \quad , \quad i=1,..,N. \tag{4.5}$$

The five terms composing the objective function (4.1) are explained as follows. The first term denotes the individual ordering cost for each item, which is incurred if and only if this item belongs to the order ($q_i > 0$). The second term denotes the purchasing costs under the assumption that the discount threshold will be achieved.

The third term reflects the expected holding cost until the next demand epoch. First note that the expected time until the next demand equals $1/\Lambda$. The expression for the expected holding cost is based on the well-known convention to shift the holding cost in a period [t+L,t+s+L] to the time interval [t,t+s] (see also Federgruen et al. (1984)). In this expression, $r_i(k)$ denotes the probability that the total demand for item i during the fixed lead time L equals k. ($r_i(k)$ can be computed recursively (see Adelson (1966).)

Finally, the fourth and fifth term represent an approximation of all future costs after the first demand epoch to come, conditioned on the event that the next occurring demand is a demand for item i (with probability λ_i/Λ) or for another item $k \neq i$ (with probability $(1-\lambda_i/\Lambda)$). Note that a demand of j units for another item $k \neq i$ causes a special replenishment opportunity for item i only if $I_k + q_k - j \leq s_k$. Hence, a demand for item k is not always a decision epoch in the single-item problem for item i. In such a case, we use $v_i(I_i + q_i, 0)$ as an approximation for the future costs of item i. (Note that the choice $v_i(I_i + q_i, 1)$ yields the same results because $v_i(I_i + q_i, 0)$ equals $v_i(I_i + q_i, 1)$ if $I_i + q_i > c_i$; see Appendix 4.1.)

The joint ordering cost A is not included in the objective function, since it does not depend on the order composition.

Condition (4.2) reflects the requirement that the new order is indeed an extension of the order generated by the can-order strategy. Condition (4.3) is introduced in order to enable the decision maker to build in a guarantee that the extended policy will not destroy completely the structure of the underlying can-order strategy. For example, one may require the maximal inventory level for item i to be bounded by $1.2 \cdot S_i$. Also limited storage capacity or technical or economical deterioration can determine the choice of UB_i. Finally, condition (4.4) indicates that the new order composition should be large enough to reach the discount threshold Q_d.

The knapsack problem **KS** can be solved by standard dynamic programming techniques. The computational complexity depends to a high extent on the feasible range of values for q_i, i.e. $[S_i - I_i, UB_i - I_i]$ for items with $I_i \leq c_i$ and $[0, UB_i - I_i]$ for items with $I_i > c_i$. (The possible values of I_i are ranging from $s_i + 1 - m_i$ to UB_i.) A greedy heuristic procedure for solving **KS** is presented in Appendix 4.2.

Recall that **KS** determines the optimal order composition under the condition that the discount is taken. Of course, also the possibility to ignore the discount opportunity and order only the initial order quantities has to be taken into account. The solution procedure is summarized in the following algorithm.

Algorithm for can-order policy with discount evaluation

Step 1: Ignoring quantity discounts compute the optimal can-order policy with the method of Federgruen et al. (1984).

Step 2: At each demand epoch at which an order is triggered according to the can-order policy from Step 1, do:

a) Determine for each item its present inventory position I_i.

b) Determine the initial order sizes: $q_i := S_i - I_i$ if $I_i \leq c_i$; $q_i := 0$ otherwise. If $\Sigma_i q_i w_i \geq Q_d$, then order q_i, $i=1,..,N$, else go to Step 2c.

c) If $\Sigma_i (UB_i - I_i) w_i < Q_d$, then order q_i, $i=1,..,N$ (discount threshold cannot be reached), else go to Step 2d.

d) Compute $C(q_1, .., q_N)$, the value of (4.1) for the vector q_i, $i=1,..,N$, with (1-d) replaced by 1. (The expected total relevant cost when the order is not enlarged.)

e) Solve **KS** yielding a vector q_i^*, $i=1,..,N$, and an objective function value, denoted by $C(q_1^*, ..., q_N^*)$.

f) If $C(q_1^*,\ldots,q_N^*) < C(q_1,\ldots,q_N)$, then enlarge the order so that the discount threshold is reached (order q_i^* of item i); else do not enlarge the initial order sizes (order q_i of item i).

We emphasize the hierarchical structure in our solution algorithm. First, the can-order strategy is determined based on information on the demand processes, the holding costs, the ordering costs, and the required service level constraint. As long as these parameters remain constant the resulting can-order policy will not change. Step 2, however, is a dynamic procedure using the actual inventory levels of all items as input parameters. So Step 2 has to be carried out on line. Hence, it is important to solve the knapsack problem **KS** (which constitutes the key element in Step 2) with a fast heuristic (such as the one presented in Appendix 4.2). The alternative, to carry out Step 2 on forehand for all possible values of the inventory vector and to store the outcome in a database, is not to be recommended.

Next, we note that the solution procedure is also hierarchical in the sense that deviating from the initial order size q_i as generated by the can-order strategy is only allowed to reach the discount threshold value. For example, one can imagine that in case the threshold value is already reached by the initial values of q_i one nevertheless obtains an improvement by changing the values of q_i. (This alternative is easily incorporated in our algorithm by a slight modification of Step 2b: proceed (always) with Step 2c after having calculated the initial order sizes.)

Finally, we note that other choices for the specific structure of the knapsack problem **KS** are possible. For example, when one wants to stick as close as possible to the original can-order strategy, it could be decided that enlargement of the total order size is only allowed by adding items which are not yet in the order and ordering such an item up to its order-up-to level S_i. The only change will then occur in the specific definition of the knapsack problem that has to be solved in Step 2e. Other discount structures as the all-units discount structure can also be handled by a redefinition of the knapsack problem (see Appendix 4.3).

4.5 Numerical examples

In this section some numerical examples are given to validate the use of the discount evaluation procedure within the framework of can-order policies. The proposed can-order policy with discount evaluation will be referred to as the *CAN⁺ system*, while the can-order strategy without discount evaluation will be referred to as the *CAN system*. Note that, under the CAN system, the discount opportunity is only used when the regular can-order replenishment is enough to qualify for the discount (the replenishment is never enlarged).

The average relevant cost per time unit of the CAN^+ system and the CAN system is compared for a family of 15 items. The relevant costs include ordering and holding costs less the savings realized by achieving a discount. The purchasing costs (without discounts) are not included because they do not depend on the inventory control system that is used. Simulation is used to find the average relevant cost per time unit because it is not possible to calculate the exact cost explicitly from the input parameters.

Table 4.1 lists the 15 items, along with the values for h_i, w_i, a_i, and λ_i. There are no shortage costs involved, but there is a service level constraint which requires that at least a fraction β of demand is satisfied directly from stock on hand. It is assumed that the demand size of all items has the same truncated negative binomial distribution with parameters $r=30$, $p=0.85$. (Hence the expected demand size equals 3.83 and the variance of the demand size equals 9.75 .) The corresponding can-order parameters s_i, c_i, and S_i are given in Table 4.1. The relative values including purchasing costs, which are needed for the discount evaluation, can then be computed by the algorithm in Appendix 4.1.

Table 4.1 Input data ($N=15$, $A=75$, $L=0.25$, $\beta=0.975$)

item i	h_i	w_i	a_i	λ_i	s_i	c_i	S_i
1	2.50	5.00	15	15	38	84	118
2	1.50	4.00	15	10	19	63	99
3	0.75	3.75	15	12	15	76	131
4	1.25	2.50	15	10	19	65	104
5	1.75	7.50	15	5	11	40	63
6	1.50	3.50	30	12	23	66	122
7	0.25	1.00	30	7	0	55	158
8	0.75	3.00	30	9	17	61	129
9	0.50	2.50	30	15	23	91	197
10	4.25	20.00	30	2	8	21	35
11	5.50	30.00	45	2	11	22	37
12	0.25	1.00	25	10	0	67	180
13	0.75	4.00	25	10	20	68	133
14	0.50	2.50	25	8	1	55	126
15	0.25	1.25	45	5	0	28	135

Table 4.2 gives the results of several experiments, where the discount threshold, Q_d, and the discount percentage, d, are varied, while the other parameters are kept fixed. For each combination of Q_d and d, the number of simulation runs is determined by the requirement that a 95%-confidence interval has to be obtained with a bandwidth of four. A single run for a given combination is obtained by simulating the multi-item system until 1000 orders have been triggered. The simulated average cost per time unit for the CAN system and the CAN^+ system is denoted by CAN and CAN^+, respectively. The percentage cost saving of using the CAN^+ system instead of the CAN system is denoted by % c.s. (% c.s. = $100 \cdot (CAN-CAN^+)/CAN$).

Table 4.2 Relevant cost per time unit for family of Table 4.1

Q_d	d	CAN	CAN^+	% c.s
1500	0.03	1357.12	1352.01	0.38
1500	0.05	1303.68	1293.05	0.82
1500	0.10	1171.21	1134.53	3.13
1500	0.20	906.14	817.59	9.77
2000	0.03	1384.81	1368.35	1.19
2000	0.05	1345.90	1315.48	2.26
2000	0.10	1254.90	1171.88	6.61
2000	0.20	1071.98	864.64	19.34
2500	0.03	1408.86	1390.13	1.33
2500	0.05	1392.07	1347.88	3.17
2500	0.10	1344.61	1215.95	9.57
2500	0.20	1252.76	929.05	25.84

Table 4.2 shows that the percentage cost saving of using the discount evaluation procedure may be considerable (up to 25%). Of course, the percentage cost saving increases as Q_d increases because the threshold value will then be reached less frequently by the regular can-order replenishment. (In our experiments, the fraction of orders for which the CAN system achieved the discount was, on the average, 0.70, 0.43 and 0.11 for Q_d = 1500, 2000, and 2500, respectively.)

4.6 Conclusions

We proposed an approach to handle discount opportunities in the framework of can-order policies. The basic can-order strategy is determined based on information of the demand processes, the holding costs, the ordering costs, the lead time, and the required service level. The can-order strategy will not change unless one of these parameters changes. At a replenishment epoch, triggered by this can-order policy, a dynamic procedure, based on the actual inventory levels, is used for the discount evaluation. This procedure decides whether or not the order has to be enlarged to achieve the discount, when the regular can-order replenishment is not large enough.

The numerical examples presented in Section 4.5 show that the proposed CAN$^+$ system performs quite satisfactorily when discount opportunities exist. The improvement over the conservative strategy (the CAN system), which only achieves the discount in case the regular order quantity is large enough, can be substantial (up to 25% of the controllable costs). In the next chapter the CAN$^+$ system will be compared with another inventory system which has been developed by Miltenburg and Silver.

Appendix 4.1 Computation of the relative values

Recall that $v_i\,(x,z)$ denotes the relative value of item i with an inventory position of x units, just after a demand event (z=0) or a special replenishment opportunity (z=1) for a fixed can-order strategy with control parameters $(S_i\,,c_i\,,s_i\,)$, i=1,..,N. The algorithm of Federgruen et al. (1984) for *the single-item problem* can be used to compute the relative values of item i. However, the purchasing cost is then neglected. In this appendix, the same approach as Federgruen et al. (1984) is used to determine the relative values of item i including purchasing cost. For convenience, the subscript i will be deleted in the notation.

For a fixed can-order strategy, with parameters (S,c,s), the average cost and relative values can be determined by the theory of regenerative processes. The attention is restricted to the cost incurred between two subsequent replenishment orders for that particular item. The regeneration state is the order-up-to level S, the state which is visited just after an order. Now, define for a can-order system, which starts in state (x,0) with x>s:

$\tau(x)$: the expected time until the next replenishment order;

$\psi(x)$: the probability that the next replenishment is triggered by a demand;

$\eta(x)$: the sum of the expected holding cost until the next replenishment and the expected purchasing cost at that particular replenishment epoch;

$\kappa(x)$: the total holding cost until the next replenishment together with the expected purchasing and ordering costs incurred at the next replenishment epoch.

It follows that

$$\kappa(x) = \eta(x) + (A+a)\psi(x) + a(1-\psi(x))\,. \tag{A.4.1}$$

The long run average cost per time unit under the can-order strategy, denoted by g(S,c,s), equals

$$g(S,c,s) = \frac{\kappa(S)}{\tau(S)}\,. \tag{A.4.2}$$

Now, $\tau(x)$, $\psi(x)$ and $\eta(x)$ are determined by conditioning on the state of the system after the next decision epoch. Recall that special replenishment opportunities occur according to a Poisson process with rate μ. Then, the probability that the next decision epoch is induced by a demand is $\lambda(\lambda+\mu)^{-1}$, whereas $\mu(\lambda+\mu)^{-1}$ equals the probability that the next decision epoch is induced by a special replenishment opportunity. Then, for $x>s$,

$$\tau(x) = (\lambda+\mu)^{-1} + \mu(\lambda+\mu)^{-1}\tau(x)\,\delta(x-c) + \lambda(\lambda+\mu)^{-1}\sum_{j=1}^{x-s-1}\tau(x-j)\,\phi(j)\,, \tag{A.4.3}$$

and

$$\psi(x) = \mu(\lambda+\mu)^{-1}\psi(x)\,\delta(x-c) + \lambda(\lambda+\mu)^{-1}\left\{\sum_{j=x-s}^{\infty}\phi(j) + \sum_{j=1}^{x-s-1}\psi(x-j)\,\phi(j)\right\}, \tag{A.4.4}$$

with $\delta(i)=1$ if $i>0$ and $\delta(i)=0$ otherwise.

Define $\omega(x)$ as the holding cost until the next decision epoch. (Federgruen et al. (1984) consider also two sorts of penalty costs, but these are disregarded in this appendix.) Using the same convention as in Section 4.4 (the holding cost in $[t+L, t+s+L]$ is assigned to the time interval $[t, t+s]$), it follows that

$$\omega(x) = h(\lambda+\mu)^{-1}\sum_{j=0}^{x}(x-j)\,r(j)\,, \tag{A.4.5}$$

where $r(j)$ denotes the distribution function of the demand during the lead time and $(\lambda+\mu)^{-1}$ is the expected time until the next decision epoch.

If at the next decision epoch a demand of j units occurs and $x-j\leq s$, then the item triggers a replenishment and an order for $(S-x+j)$ units is placed. If the next decision epoch is a special replenishment opportunity and $x\leq c$, then the item is also included in the replenishment and a purchasing cost of $(S-x)\,w$ dollars is incurred. Hence, for $x>s$:

$$\eta(x) = \omega(x) + \mu(\lambda+\mu)^{-1}\eta(x)\,\delta(x-c) + \lambda(\lambda+\mu)^{-1}\sum_{j=1}^{x-s-1}\eta(x-j)\,\phi(j) + \mu(\lambda+\mu)^{-1}(S-x)\,w\,\delta(c+1-x) + \lambda(\lambda+\mu)^{-1}\sum_{j=x-s}^{\infty}(S-x+j)\,w\,\phi(j)\,, \tag{A.4.6}$$

where w is average purchasing cost (*after discounts*) which has to be paid when using the CAN$^+$ system. So, after every order, w is updated and the relative values are recalculated.

Finally, the relative values of the given can-order strategy are defined by

$$v(x,0) = \begin{cases} \kappa(x) - g(S,c,s)\,\tau(x) & x > s\,, \\ A + a + (S-x)w & x \le s\,, \end{cases} \tag{A.4.7}$$

$$v(x,1) = \begin{cases} \kappa(x) - g(S,c,s)\,\tau(x) & x > c\,, \\ a + (S-x)w & x \le c\,. \end{cases} \tag{A.4.8}$$

Algorithm for determining v(x,z)

Step 1: Compute $\tau(x)$, $\psi(x)$, $\eta(x)$, $\kappa(x)$ recursively from (A.4.3), (A.4.4), (A.4.5), (A.4.6), and (A.4.1) for $x=s+1,..,UB$.

Step 2: Compute $g(S,c,s)$ from (A.4.2).

Step 3: Compute $v(x,z)$ for all relevant values of x and z from (A.4.7) and (A.4.8).

Appendix 4.2 A heuristic approach to solve the knapsack problem KS

The knapsack problem **KS**, which has been presented in Section 4.4, can be solved by standard dynamic programming techniques. However, such an approach may be rather time consuming due to the large range of the feasible values of q_i ,$i=1,..,N$. Besides, the objective function (4.1) is only an approximation for the sum of the direct costs and the future costs. For these two reasons, a heuristic solution procedure is used in the CAN$^+$ system to solve the knapsack problem.

Recall that the (separable) objective function

$$C(q_1,..,q_N) = \sum_{i=1}^{N} C_i(q_i)\,, \tag{A.4.9}$$

with

$$C_i(q_i) = \delta(q_i)\,a_i + (1-d)\,q_i w_i + \frac{h_i}{\Lambda}\sum_{k=0}^{I_i+q_i}(I_i+q_i-k)\,r_i(k) + \tag{A.4.10}$$
$$\frac{\lambda_i}{\Lambda}\sum_{j=1}^{m_i}\phi_i(j)\,v_i(I_i+q_i-j,0) + \left(1-\frac{\lambda_i}{\Lambda}\right)v_i(I_i+q_i,0)\ ,$$

has to be minimized under the condition that the discount threshold is achieved and each individual order quantity satisfies $L_i \le q_i \le U_i$, where $U_i = UB_i - I_i$, $L_i = S_i - I_i$ if $I_i \le c_i$ and $L_i = 0$ otherwise. (Note that the knapsack problem has always a feasible solution because of the feasibility check in Step 2c of the discount evaluation procedure in Section 4.4.) The algorithm to find the optimal order quantities is summarized below.

Algorithm to solve KS

Step 1: Determine $q_i = arg\ min_{L_i \le x \le U_i} C_i(x)$ with (A.4.10) for all i (q_i is the individual optimal order quantity of item i under the condition that the discount is achieved for the family).

Step 2: If $\Sigma_i q_i w_i \ge Q_d$ then $q_i^* := q_i$, $i=1,..,N$ (solution is feasible), else go to Step 3.

Step 3: Make the solution feasible:

a) Determine Δ_i, the increase of the total cost per added dollar value, for each item i ($i=1,..,N$).

$$\Delta_i = \begin{cases} \dfrac{C_i(y_i) - C_i(0)}{y_i w_i} & if\ \ q_i = 0\ , \\[2ex] \dfrac{C_i(q_i+1) - C_i(q_i)}{w_i} & if\ \ 0 < q_i < U_i\ , \\[2ex] \infty & if\ \ q_i = U_i\ , \end{cases} \tag{A.4.11}$$

where $y_i = arg\ min_{0 < x \le U_i} C_i(x)$.

b) Repeat until $\Sigma_i q_i w_i \ge Q_d$:

- Determine $j := arg\ min_i \Delta_i$;
- If $q_j > 0$ then $q_j := q_j + 1$; else $q_j := q_j + y_j$;

- Recalculate Δ_j with (A.4.11).

c) $q_i^* = q_i$ for $i=1,..,N$.

Step 3 is needed when the individual optimal order quantities obtained in Step 1 are not large enough to achieve the discount. In this case, an iterative procedure is used, where in every iteration one unit is added of the item which causes the smallest cost increase of the objective function per added dollar value. However, when $q_i = 0$, it is better to increase the order with y_i units instead of one unit. It seems reasonable that $C_i(x)$ is convex in x, with a discontinuity in $x=0$, due to the factor $\delta(x)a_i$ in (A.4.10). Hence, if item i is included in the order, the best order quantity for item i is y_i. To avoid violations of condition (4.3), Δ_i is set equal to infinity if $q_i = U_i$.

Appendix 4.3 Discount evaluation for other discount structures

The traditional quantity discount models analyze unit price discounts. In this research we investigate one type of unit price discounts, namely all-units discounts.

Another well-known unit-price discount structure is an *incremental-units discount*, where a discount of d percent is given on the total dollar value *above* the threshold value Q_d (i.e. the actual purchasing cost of an order of Q dollars is $Q_d+(1-d)(Q-Q_d)$ if $Q \geq Q_d$ and Q otherwise). The discount evaluation procedure presented in Section 4.4 can be used for this situation after the following (slight) modifications:

- In Step 2b: proceed directly with Step 2c after the determination of the initial order quantities. (Because discounts are only achieved on the order value above the threshold.)
- In Step 2d: compute the value $C(q_1,..,q_N)$ of (4.1) for the vector q_i, $i=1,..,N$, with (1-d) replaced by 1 if $\Sigma_i q_i w_i < Q_d$; else compute the value $C(q_1,..,q_N)$ of (4.1) for the vector q_i, $i=1,..,N$, and add a cost factor $d \cdot Q_d$ to the cost obtained by formula (4.1) to account for the fact that discounts are only achieved on the order value above the threshold Q_d.
- In Step 2e: add a cost factor $d \cdot Q_d$ to the cost obtained by formula (4.1).

Another type of discount structure, which exists in many practical situations, is a *fixed dollar value discount on the ordering cost or freight cost* when the dollar value of the replenishment order exceeds the discount breakpoint. A typical example is the situation where FR:=0 if $Q \geq Q_d$, and FR:=F if $Q < Q_d$, where FR denotes the freight cost per replenishment. The discount evaluation procedure of Section 4.4 is also applicable to the freight cost discount structure after the following changes:

- In the objective function (4.1) of **KS**: set the discount percentage d equal to zero.
- In Step 2d: add a cost factor F to $C(q_1,..,q_N)$ which is calculated with (the adapted) formula (4.1) (because $\Sigma_i q_i w_i < Q_d$).

The joint ordering cost that is used to calculate the basic can-order policy is equal to A+F in the freight cost discount case. Further, we recommend to approximate the joint ordering cost by A+F' (with F' the average paid freight cost) and to set the purchasing cost w equal to zero, when calculating the relative values as in Appendix 4.1.

So far, we only considered discount structures with a single threshold value. Frequently, a discount schedule consists of different discount rates on different ranges of the dollar value. For example, a discount of d_j % is offered on the total dollar value Q if $Q_j < Q < Q_{j+1}$, j=0,..,M. Such a discount structure may be handled within the framework of the CAN$^+$ system by considering different knapsack problems for different ranges of Q (knapsack problem $\mathbf{KS}_j$ is then the same as **KS** with $Q_d = Q_j$ and $d = d_j$).

Discount evaluation with multiple discount rates

Step 1: Ignoring quantity discounts compute the optimal can-order policy with the method of Federgruen et al. (1984).

Step 2: At each demand epoch at which an order is triggered according to the can-order policy from Step 1:

a) Determine for each item its present inventory level I_i.

b) Determine the initial order sizes: $q_i := S_i - I_i$ if $I_i \leq c_i$; else $q_i := 0$. Determine (range) r for which $Q_r \leq \Sigma_i q_i w_i < Q_{r+1}$. If r=M then order q_i, i=1,..,N.

c) Determine the minimum value of (range) s for which $\Sigma_i (UB_i - I_i) w_i < Q_s$. If s=r+1, then order q_i, i=1,..,N (next discount threshold cannot be reached); else go to Step 2d.

d) Compute $C(q_1,..,q_N)$, the value of (4.1) for the vector q_i, $i=1,..,N$, with $(1\text{-}d)$ replaced by $(1\text{-}d_r)$. (The expected total relevant costs when the order is not enlarged.)

e) For $j=r+1,..,s\text{-}1$ do:
Solve $\mathbf{KS}_j$ yielding a vector q_i^*, $i=1,..,N$, and the objective function value $C_j(q_1^*,...,q_N^*)$.

f) If $\min_j C_j(q_1^*,...,q_N^*) < C(q_1,...,q_N)$, then enlarge the order so that the discount threshold $\arg\min_j C_j(q_1^*,...,q_N^*)$ is reached (order the corresponding q_i^* of item i); else do not enlarge the initial order sizes (order q_i of item i).

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CHAPTER 5

COMPARISON OF THE CAN$^+$ SYSTEM WITH THE MILTENBURG SYSTEM

This chapter considers two inventory control systems, that both take into account joint ordering costs and discount opportunities: the CAN$^+$ system, which is introduced in Chapter 4, and the Miltenburg system. The performance of both systems is compared both from a qualitative and a quantitative point of view. To enable a quantitative comparison an adapted version of the Miltenburg system is developed for the case of simple Poisson demands.

5.1 Introduction

In the previous chapter we considered a multi-item inventory problem where joint replenishments of different items may lead to cost savings caused by reduced joint ordering costs and (all-units) discounts. An algorithm has been developed to handle discount opportunities within the framework of can-order policies. The can-order policy, which ignores quantity discounts, is used as a basic ordering strategy. At an epoch at which an order is triggered according to the can-order system, the composition of the order is determined via a one period look ahead rule, which incorporates the potential for exploiting the quantity discount. This system, which has been described in detail in Chapter 4, will be referred to as the *CAN$^+$ system*.

Another inventory control system, that also accounts for joint ordering costs and quantity discounts, has been developed by Miltenburg and Silver. They have shown that their system, which will be referred to as the *Miltenburg system*, outperforms other, more well-known, inventory control systems, like IBM's IMPACT (see IBM (1971) for more details on IMPACT).

Section 5.4 and 5.5 are based on the paper "Coordinated replenishment systems with discount opportunities" (co-authors: F.A. van der Duyn Schouten and R.M.J. Heuts), which has been submitted for publication to *International Journal of Production Research*.

The objective of this chapter is to compare the performance of the Miltenburg system with that of the CAN$^+$ system both from a qualitative and a quantitative point of view. However, an empirical quantitative comparison is complicated by the different demand assumptions (demand processes in Miltenburg and Silver's system are described by a Wiener process, whereas the CAN$^+$ system uses compound Poisson processes). Moreover, several details of the Miltenburg system are missing in the available literature, which complicates the implementation into a computer program. For these two reasons, we decided to adapt the Miltenburg system for simple Poisson demand processes.

This chapter is organized as follows. We start, in Section 5.2, with a description of the Miltenburg system. Some differences with the CAN$^+$ system are discussed. The adapted version is described in Section 5.3. Section 5.4 gives the details of the empirical comparison of the Miltenburg system with the CAN$^+$ system under Poisson demands. The conclusions of the comparative study are summarized in Section 5.5.

5.2 Description of the Miltenburg system

The Miltenburg system, which is suitable for both continuous review and periodic review, uses a reorder point concept. When the inventory position of any item drops (to or) below its reorder point at a review instant, then a family replenishment is triggered. Based on the actual inventory position of all items, the relevant costs, and the discount structure, a group order quantity is selected. This group order quantity is obtained by aggregating information about all items into one single item. The discount proposal is evaluated by using a deterministic model of Brown (1967).

The group order quantity is then allocated among the items in the family in such a way that the expected time until the next replenishment is maximized. Miltenburg and Silver model the demand for individual items as independent diffusion processes (a diffusion process has the attractive properties that total demand over any given interval of time has a normal distribution and that its sample paths are continuous functions of time).

The determination of the probability distribution function of the residual stock at the next trigger moment is an important issue when establishing the reorder points. The residual stock of an item is the excess stock above the reorder point when a family order is triggered. Neglecting this residual stock would imply that more safety stock is provided than necessary. The probability distribution function of the residual stock at the next trigger moment depends on the inventory position of all the items after the allocation.

Thus, after every allocation these distributions are evaluated and used to recalculate the reorder points.

Summarizing, at a trigger moment, a group order quantity for the entire family is selected using Brown's model. Then, this order quantity is allocated among the items in the family and, finally, based on the inventory positions after the allocation the reorder points are recalculated. In a sequence of papers, Miltenburg (1985), and Miltenburg and Silver (1984a,b,c) have provided procedures for each of these steps.

It is known that the joint replenishment problem with joint ordering costs and discounts is a very complex problem. Heuristic rules such as used in the system of Miltenburg are therefore unavoidable. (For example, the sequential approach of the three decisions to be made.) Nevertheless, some critical notes can be made:

- The reorder points are recalculated after every order, even when the parameters for the demand process, the lead time process, and the service level constraint remain constant in subsequent order cycles. This may complicate the acceptance of the system by the inventory planner.
- All discount structures have to be converted to a standard form: the all-units discount structure. (Here, the Miltenburg system uses the same approach as IMPACT; see IBM (1971).) Such a conversion may lead to inaccuracies. Different discount structures can be handled more easily in the framework of the CAN^+ system (see Appendix 4.3).
- Miltenburg models the inventory position of an item as a diffusion process drifting from some starting inventory at time zero towards an absorption barrier at the reorder point. This implies that, under continuous review, no undershoots of the reorder point can occur. This may be too restrictive in some situations. (The CAN^+ system takes account of undershoots of the reorder point.)
- Due to the approximations used in the Miltenburg system the required service level can not be guaranteed. (In Section 5.4 it will be shown that substantial deviations of the required service level may occur.)
- The computations which have to be made by the Miltenburg system at every trigger moment may require a large amount of computation time. To lower this computation time, different (less effective) procedures have to be used when the system is implemented on a microcomputer (see Miltenburg and Silver (1988)).

A detailed description of the alternative multi-item system, the CAN^+ system, has been given in Chapter 4. The major drawbacks of this system are:

- The CAN^+ system ignores the existence of discount opportunities when setting the parameters of the *basic* can-order strategy. Since large deviations from the basic can-order replenishment will be avoided, it may happen that some promising discount proposals are neglected because they are not achievable.
- The CAN^+ system provides more service than specified. The parameters of the basic can-order strategy are set such that the service level constraint will be satisfied, while the actual service level will increase due to the enlargements of the basic can-order replenishments.

In general, it is not easy to choose between both inventory control systems based on the above qualitative comparisons only. A quantitative comparison, based on simulation, can provide some additional support. However, such a comparison is not straightforward due to the different demand assumptions. Although these assumptions seem to be of a rather technical nature it should be admitted that the computational procedures in both approaches depend quite heavily on these assumptions. In order to compare both approaches we present in the next section a version of Miltenburg and Silver's inventory control system for the case of (simple) Poisson demand. We have not been able to adapt these procedures for the compound Poisson demand case except by approximating compound Poisson processes by simple Poisson processes. On the other hand, adaption of the CAN^+ system to the case of Wiener demand processes is not straightforward either.

Of course, simple Poisson processes are not a better description of empirical demand processes. In general, a compound Poisson process or a diffusion process is more realistic. However, for Poisson demands, the structure of the original Miltenburg system is well preserved, which allows a fair numerical comparison of the Miltenburg system with the CAN^+ system.

5.3 Adapted version of the Miltenburg system

In this section we present the adapted version of the Miltenburg system for the Poisson demand case. Recall that three decisions have to be made at the moment any item in the family hits its reorder point: (i) determination of the group order quantity; (ii) allocation of the group order quantity among the individual items; and (iii) recalculation of the reorder points. Although the first decision is not affected by the demand distribution (a deterministic model is used for the evaluation), it will also be discussed for the sake of completeness. Attention will be focused on the situation with an all-units discount structure and continuous review.

The following symbols will be used in the next sections:

N : number of items in the family;
L : lead time (equal for all items);
β : required fraction of demand which is satisfied directly from stock on hand;
Q : group order quantity (in dollars);
Q_d : discount threshold (in dollars);
d : discount percentage;
A : joint ordering cost per replenishment;
a_i : individual ordering cost per replenishment for item i (i=1,..,N);
w_i : unit purchasing cost of item i (i=1,..,N);
h_i : holding cost of item i per unit per unit time (i=1,..,N);
r : holding cost rate per dollar ($r:=h_i/w_i$ is equal for all items i);
λ_i : Poisson arrival rate of customers for item i (i=1,..,N);
s_i : reorder point of item i (i=1,..,N);
J_i : residual stock of item i at a trigger moment (i.e. inventory position minus the reorder point of item i) (i=1,..,N);
Δ_i : residual stock of item i just after the allocation of the group order quantity (i=1,..,N);
q_i : order quantity of item i (in units) (i=1,..,N).

5.3.1 Determination of the group order quantity

The determination of the group order quantity is complicated because it affects not only the (direct) costs at the current replenishment epoch, but also future decisions (and hence future costs). Miltenburg and Silver (1984c) have developed a probabilistic approach which accounts for these future effects, but they conclude that the error of using other, deterministic, models is small. The discount evaluation model used in the Miltenburg system is therefore based on Brown's net present value analysis (see Brown (1967)).

Brown's model focuses on the situation where a one-change opportunity is offered to purchase a quantity Q_0 of a single item at a reduced price. (This item has a deterministic demand and a positive inventory level.) Q_0 is chosen such that the net present value of the current and future costs is minimized. First, note that this discount structure is different from the all-units discount structure in the sense that the discount applies for all values of Q_0 in the Brown model, whereas under the all-units discount structure the discount is only obtained if Q exceeds the given threshold. Further, Brown's model refers to a single-item problem while the coordinated control system attempts to find an order quantity for a family of items with different demand rates and different unit purchasing costs. The order quantity in Brown's model is expressed in units. However, an order quantity in the multi-item case should be expressed in dollars, due to the different unit costs (w_i) of the items.

Miltenburg proposes to create an aggregate item with demand per unit time $D_0 = \Sigma_i \lambda_i$, unit purchasing cost $w_0 = (\Sigma_i \lambda_i w_i)/D_0$, ordering cost $A_0 = A + \Sigma_i a_i$ and inventory position $J_0 = \Sigma_i J_i$. (Note that not the actual inventory level of item i is used, but its residual stock.) The breakpoint quantity, Q_d (in dollars), is converted to a breakpoint quantity $Q_b = Q_d / w_0$ (in units). The problem then reduces to find the order quantity Q_0 (in units) that minimizes

$$NPV(Q_0) := (A_0 + Q_0 w_0 (1 - \delta(Q_0))) + (A_0 + Q_{ND} w_0 + \frac{D_0 w_0}{r}) e^{\frac{-r(Q_0 + J_0)}{D_0}}, \tag{5.1}$$

where $\delta(Q_0) = 0$ if $Q_0 < Q_b$
$\delta(Q_0) = d$ if $Q_0 \geq Q_b$,

and $$Q_{ND} = \sqrt{\frac{2 A_0 D_0}{w_0 r}}. \tag{5.2}$$

It is shown in Miltenburg and Silver (1984c) that the optimal order quantity, Q_0, is equal to

$$Q_1 := \frac{D_0}{r} \ln \left\{ \frac{r}{D_0 w_0 (1-d)} \left(\frac{D_0 w_0}{r} + A_0 + Q_{ND} w_0 \right) \right\} - J_0 \tag{5.3}$$

if $Q_1 \geq Q_b$. Otherwise, either $Q_0 = Q_{ND} - J_0$ or $Q_0 = Q_b$ is selected, depending on which one yields a lower net present value.

The group order quantity, Q, is then given by $Q_0 w_0$. We add a slight improvement to this procedure. Note that in case $Q_0 = Q_1$ or $Q_0 = Q_{ND} - J_0$, an amount of $J_0 w_0$ dollars is subtracted from the order quantity to account for the current residual stock. This term is an approximation in the multi-item case for the actual residual stock level $\Sigma_i J_i w_i$.

Therefore, $J_0 = (\Sigma_i J_i w_i)/w_0$ is used instead of $J_0 = \Sigma_i J_i$ in the discount evaluation algorithm.

A problem arises when Q_0 is negative. This situation may occur if a number of items have a high residual stock at the moment an order is triggered. In our computer program we chose (arbitrarily) to select $Q_0 = Q_{ND}$ in such a case. (Note that the choice $Q_0 = 0$ will bring us into a loop.)

The algorithm for the determination of the group order quantity is outlined below.

Algorithm for determination of the group order quantity

Step 1: (Transformation of the multi-item problem into a single-item problem)

$D_0 := \Sigma_i \lambda_i$, $w_0 := (\Sigma_i \lambda_i w_i)/D_0$, $A_0 := A + \Sigma_i a_i$, $J_0 := (\Sigma_i J_i w_i)/w_0$, $Q_b := Q_d / w_0$.

Calculate Q_{ND} from (5.2).

Step 2: (Determination of Q_0)

Calculate Q_1 from (5.3).

If $Q_1 \geq Q_b$ then $Q_0 := Q_1$; otherwise:

- Calculate $NPV(Q_{ND} - J_0)$ and $NPV(Q_b)$ from (5.1).
- If $NPV(Q_{ND} - J_0) < NPV(Q_b)$ then $Q_0 := Q_{ND} - J_0$; else $Q_0 := Q_b$.

Step 3: (Determination of Q)

If $Q_0 > 0$ then $Q := Q_0 w_0$; otherwise $Q := Q_{ND} w_0$.

Finally, note that, by defining $A_0 = A + \Sigma_i a_i$, it is implicitly assumed that all items in the family will be included in every replenishment. In the Miltenburg system, it is possible to replenish certain items (usually low usage items) only once in every two, three, or more cycles (these items are called *multiple-cycle items*). This possibility has not been implemented in the adapted version of the Miltenburg system .

5.3.2 *Allocation of the group order quantity among the items*

Once the total order quantity has been set, this order has to be allocated among the items. Suppose that a certain item has triggered an order and a group order quantity Q (in dollars) has been selected. Denote the stock in excess of the reorder point *after the allocation* by Δ_i (in units). Then the problem reduces to the determination of the optimal mix such that $\Sigma_i (\Delta_i - J_i) w_i = Q$. The objective used in Miltenburg (1985) to select the vector $(\Delta_1, .., \Delta_N)$ is to maximize the expected time until the next reorder.

The most straightforward policy would be to allocate in such a way that Δ_i / λ_i is equal for all items i. However, this so-called ERT rule (Equalization of Run out Times) gives incorrect allocations if the group order quantity Q is small. Miltenburg (1985) has suggested another method to find the optimal allocation when demands are modeled as diffusion processes.

For Poisson demand, a similar problem has already been solved by Low and Waddington (1967). The situation in Low and Waddington differs from ours in the sense that in their problem the quantity to be allocated among the items is expressed in units (hence, $w_i = 1$ for all items i). Low and Waddington first develop an exact method which is rather time consuming. Therefore, they suggest a fast, but inexact, *interpolation method*, that is based on properties of the optimal solution. This interpolation method is used in our adapted version of the Miltenburg system. To account for different unit costs, an arrival rate $\lambda_i w_i$ is used for item i (instead of λ_i). Hence, the interpolation method selects the optimal allocation vector $(x_1, .., x_N)$ such that $\Sigma_i x_i := Q + \Sigma_i J_i w_i$, where x_i denotes the stock of item i in excess of the reorder point, *in dollars*, after the allocation. The interpolation method is outlined below. (For details, the reader is referred to the paper of Low and Waddington (1967).)

Algorithm to find the optimal allocation $(x_1, .., x_N)$ of a dollar amount X

Step 1: Calculate

$$c_1 := -0.941 + 0.625\,N,\quad c_2 := -0.538 + 0.500\,N,\quad c_3 := +\,1.478 - 0.125\,N.$$

Step 2: Calculate

$$g_1 = \frac{X + 4.036\,c_2 + 21.761\,c_3}{c_1 + 3.571\,c_2 + 14.807\,c_3},$$

$$g_2 = 3.571\,g_1 - 4.036\ ,\quad g_3 = 14.807\,g_1 - 21.761\ .$$

Step 3: Calculate $\theta_i = \lambda_i w_i \Big/ \sum_{j=1}^{N} \lambda_j w_j$.

Step 4: Determine for all items i:

$$x_i = \begin{cases} g_2 + \dfrac{g_2 - g_1}{0.12}(\theta_i - 0.15) & \textit{if}\ \ \theta_i \le 0.15\ , \\[2ex] g_2 + \dfrac{g_3 - g_2}{0.60}(\theta_i - 0.15) & \textit{if}\ \ \theta_i > 0.15\ . \end{cases}$$

(The coefficients c_1 and c_2 are mistyped in the original paper of Low and Waddington.)

Although the interpolation method is an efficient tool to solve the allocation problem, there remain some problems:

- The outcome of the interpolation method, $\Sigma_i x_i$, usually differs slightly from the desired total $Q + \Sigma_i J_i w_i$. (This problem was already mentioned by Low and Waddington in their paper.) The difference is allocated to each item i proportional to its mean dollar demand rate ($\lambda_i w_i$).
- The interpolation method allocates $Q + \Sigma_i J_i w_i$ over the items. It may happen that an allocation with $x_i < J_i w_i$ is suggested. This implies that the inventory position of item i after the allocation is smaller than before the allocation. Neither Miltenburg, nor Low and Waddington mention this problem. We handle this situation as follows. Let E denote the set of items with $x_i \leq J_i w_i$ after a run of the interpolation method. For all items i in E, x_i is set equal to $J_i w_i$ and the interpolation method is repeated with $Q + \Sigma_{i \notin E} J_i w_i$ to obtain the allocation for all items that are not included in E. This procedure is repeated until there is no item i with $x_i < J_i w_i$.
- Since the allocation vector $(x_1, .., x_N)$ is expressed in dollars, the order quantity of item i, $x_i - J_i w_i$ (in dollars), has to be converted to an order quantity q_i (in units). A straightforward approach is to round $(x_i - J_i w_i)/w_i$ to the nearest integer q_i. However, in general, this will cause a difference between the actual group order quantity $\Sigma_i q_i w_i$ and the desired quantity Q. The difference may be considerable if some unit purchasing costs are high. Our implementation of the Miltenburg system attempts to decrease the difference by changing the rounding of some items. An even more important problem occurs when $Q \geq Q_d$ while $\Sigma_i q_i w_i < Q_d$. This may occur, in particular, when $Q = Q_d$. This problem is solved by increasing the order quantities q_i of some items i by one unit, such that $\Sigma_i q_i w_i \geq Q_d$.

Another, more time consuming, allocation procedure would be the incremental solution technique which has also been used by Miltenburg (1985). A key element in this procedure is the calculation of the expected time until the next order for a given allocation $(\Delta_1, .., \Delta_N)$. Low and Waddington (1967) give an expression for this expected cycle time for Poisson demands.

5.3.3 Calculation of the reorder points

Finally, after the allocation of the family order among the individual items, the reorder points must be set for each item. At the selection of the reorder points, the probability distribution of the residual stock level (at the following reorder moment) has to be taken into account, because this extra stock has the effect of increasing the effective safety stock.

Recall that Δ_i denotes the excess stock above the reorder point of item i, just after the allocation ($\Delta_i = J_i + q_i$). Define T_i as the time which elapses until Δ_i units have been demanded of item i. Because demand for item i is generated by an independent Poisson process with rate λ_i, it follows that T_i is Erlang distributed with parameters λ_i and Δ_i. In Appendix 5.1 it is shown that the probability distribution function, $\Phi_i(k)$, of the residual stock for item i at the next trigger moment equals

$$\Phi_i(k) = \begin{cases} \displaystyle\int_{t=0}^{\infty} f_i(t) \prod_{j \neq i} (1 - F_j(t))\, dt & \text{for } k=0\ , \\ \displaystyle\int_{t=0}^{\infty} \frac{(\lambda_i t)^{(\Delta_i - k)}}{(\Delta_i - k)!} e^{-\lambda_i t} f^{(-i)}(t)\, dt & \text{for } k=1,..,\Delta_i\ , \end{cases} \tag{5.4}$$

where

$$f^{(-i)}(t) = \sum_{j \neq i} f_j(t) \prod_{k \neq i,j} (1 - F_k(t))\ , \tag{5.5}$$

and where $f_i(t)$ and $F_i(t)$ denote the probability density function and the distribution function, respectively, of T_i.

The probability distribution function $\Phi_i(k)$ plays an important role in the calculation of the reorder points. Recall that a fraction β of demand has to be satisfied directly from stock on hand. Miltenburg and Silver approximate the expected fraction of demand for item i which is backordered by ES_i / EQ_i, where ES_i denotes the expected number of stock outs of item i during the next order cycle (which starts as soon as the current order arrives), and EQ_i denotes the expected demand during an order cycle. In the Miltenburg system, EQ_i is estimated by historical data. The expected number of stock outs during the next order cycle is equal to

$$ES_i = \sum_{k=0}^{\Delta_i} \Phi_i(k) \sum_{j=s_i+k}^{\infty} (j-s_i-k)\frac{(\lambda_i L)^j}{j!}e^{-\lambda_i L} - \sum_{j=s_i+\Delta_i}^{\infty} (j-s_i-\Delta_i)\frac{(\lambda_i L)^j}{j!}e^{-\lambda_i L}. \quad (5.6)$$

Algorithm to determine s_i given the vector $\Delta:=(\Delta_1,..,\Delta_N)$

Step 1: Determine the probability function $\Phi_i(k)$, $k=0,..,\Delta_i$ from (5.4) and (5.5).

Step 2: a) Initialize $s_i:=0$.

b) Calculate ES_i from (5.6).

c) Stop if $ES_i < (1-\beta)EQ_i$; otherwise increase s_i by one unit and return to Step 2b.

This completes the description of the modification of the Miltenburg system for Poisson demand.

5.4 Empirical comparison

The procedure developed in the previous section is used to compare numerically the performance of the Miltenburg system with the CAN$^+$ system. The performance measures are the total relevant cost per time unit and the actual realized service. The total relevant cost includes ordering and holding cost under subtraction of the savings realized by achieving a discount. The purchasing cost is not included because it does not depend on the inventory control system which is used. For different problem settings, we obtain a 95% confidence interval (with a bandwidth of one) for the total cost per time unit by repeating a number of simulation runs of 1000 order cycles. In the tables below, the average relevant cost of the CAN$^+$ system and the Miltenburg system will be denoted by CAN$^+$ and MIL, respectively.

Table 5.1 shows a set of test examples. Family 1 is a set of five identical products with the following parameters:

$$a_i=2,\ h_i=0.25,\ w_i=1,\ \lambda_i=10.$$

Family 2 consists of ten identical products with

$$a_i=1,\ h_i=0.25,\ w_i=1,\ \lambda_i=5.$$

For both families, the ordering cost ratio (i.e. the ratio of the joint ordering cost A and the average individual ordering cost $\bar{a}$) varies from 0 to 20. Table 5.1 lists the total cost per time unit for a fixed value of L, β and d, and different values of Q_d.

Table 5.1 Relevant cost per time unit (L=0.25 , β=0.97, d=0.10)

$A/\bar{a}$	Q_d	family 1 (N=5)		family 2 (N=10)	
		CAN^+	MIL	CAN^+	MIL
0	40	15.5	17.1	19.2	23.1
0	60	16.5	17.1	19.2	23.1
0	100	17.2	19.4	19.4	26.0
2	40	19.8	19.2	20.0	22.5
2	60	20.5	19.2	22.2	22.5
2	100	21.1	20.8	24.7	24.6
20	40	38.0	35.4	35.7	33.2
20	60	38.0	35.4	35.7	33.2
20	100	38.0	35.4	35.7	33.2

It appears that the relative performance of the two control systems is quite sensitive with respect to the ordering cost ratio. The CAN^+ system outperforms the system of Miltenburg for small ratios, whereas the opposite happens for large ordering cost ratios. The performance is comparable for moderate ratios (like two). The bad performance of CAN^+ for large ordering cost ratios is probably due to the method by which the basic can-order strategy is determined (Federgruen et al. (1984)). As will be shown in Chapter 6, the decomposition approach of Federgruen et al. leads to inaccurate results if the ordering cost ratio is large. This not only affects the parameters of the basic strategy but also the relative values which play an important role in the trade-off between savings by discounts and extra holding costs, when enlarging the regular can-order replenishment. (For large ratios, an alternative for the CAN^+ system is the RS^+ system which will be discussed in Chapter 7.) The actual service level, the second performance measure for the control systems, in all experiments is greater than or equal to the required fraction β.

The experiments are also run with families of products with nonidentical values of a_i, h_i, w_i, and λ_i (i=1,..,N). These values are given in Table 5.2. Note that $\bar{a}$ and the product $w_i \lambda_i$ for all items i are the same as in the experiments with the identical products. Table 5.3 shows that essentially the same results are obtained as in the test examples with the identical products.

Table 5.2 Input data for experiments with nonidentical products

family 3 (N=5)				family 4 (N=10)			
a_i	h_i	w_i	λ_i	a_i	h_i	w_i	λ_i
1.0	0.50	2.0	5.0	1.0	0.25	1.00	5.00
1.0	0.25	1.0	10.0	1.0	0.25	1.00	5.00
2.0	0.13	0.5	20.0	1.0	0.20	0.80	6.25
3.0	0.25	1.0	10.0	1.0	0.20	0.80	6.25
3.0	0.50	2.0	5.0	1.0	0.15	0.60	8.33
				1.0	0.15	0.60	8.33
				1.0	0.10	0.40	12.50
				1.0	0.10	0.40	12.50
				1.0	0.05	0.20	25.00
				1.0	0.05	0.20	25.00

Table 5.3 Relevant cost per time unit ($L=0.25$, $\beta=0.97$, $d=0.10$)

$A/\bar{a}$	Q_d	family 3 (N=5)		family 4 (N=10)	
		CAN^+	MIL	CAN^+	MIL
0	40	15.8	18.4	17.6	20.1
0	60	17.0	18.5	17.8	20.2
0	100	17.3	20.3	17.8	22.6
2	40	19.6	19.8	18.8	19.5
2	60	20.4	19.8	20.2	19.6
2	100	22.0	21.3	22.6	21.4
20	40	39.1	36.6	33.1	29.8
20	60	39.1	36.6	33.1	29.8
20	100	39.1	36.7	33.1	29.9

The effect of the lead time has been investigated by running the same experiments for family 1 and 2 with a lead time of 1 and 5 time units (instead of 0.25). The relative performance of the two control inventory systems is not really affected by the longer lead time. However, the system of Miltenburg provides less service than specified when the ordering cost ratio is large ($A/\bar{a}=20$). Hence, it seems that the Miltenburg system does not satisfy the service level constraint in situations with a large ordering cost ratio and long lead times.

Table 5.4 shows the results of some test examples to determine the effect of the required service level β. It appears that the relative performance of the inventory control systems is rather insensitive to the value of β.

Table 5.4 Relevant cost per time unit (family 1, L = 0.25, Q_d = 60 , d= 0.10)

$A/\bar{a}$	β	CAN$^+$	MIL	MIL − CAN$^+$
0	0.900	14.1	14.7	+0.6
0	0.970	16.5	17.1	+0.6
0	0.995	20.4	20.9	+0.5
2	0.900	17.9	16.7	- 1.2
2	0.970	20.5	19.2	- 1.3
2	0.995	24.1	22.8	- 1.3
20	0.900	38.0	35.2	- 2.8
20	0.970	38.0	35.4	- 2.6
20	0.995	42.5	39.1	- 3.4

In all experiments, the maximal inventory position of item i has been restricted to $1.2 \cdot S_i$ when using the CAN$^+$ system. This provides an upper bound on the order quantities in the CAN$^+$ system. This restriction, which is not incorporated in the Miltenburg system, may cause that some discount breakpoints are not attainable for the CAN$^+$ system. Apart from this point it follows from the experiments that the CAN$^+$ system is more conservative in taking a discount than the Miltenburg system. While the Miltenburg system realizes, relative to the CAN$^+$ system, more savings by taking discounts, the CAN$^+$ system has lower total ordering and holding cost, at least for small and moderate ordering cost ratios. So, it may be expected that the relative performance of the Miltenburg system improves if the potential savings from the discount increase. Table 5.5 shows the results of some test examples where the potential savings, due to discounts, are varied by changing the discount percentage d. The results confirm our conjecture about the improved relative performance of Miltenburg's system, although it is not always true (see also Table 5.7).

Table 5.5 Relevant cost per time unit ($A/\bar{a}=2$, $L=0.25$, $\beta=0.97$, $Q_d=60$)

d	family 1			family 2		
	CAN^+	MIL	MIL − CAN^+	CAN^+	MIL	MIL − CAN^+
0.01	24.2	23.9	- 0.3	24.9	28.9	+4.0
0.05	22.6	21.6	- 1.0	24.0	24.9	+0.9
0.10	20.4	19.1	- 1.3	22.2	22.5	+0.3
0.20	15.7	15.1	- 0.6	18.6	18.4	- 0.2

We do not pretend that the comparison is exhaustive. The (relative) performance of both systems is influenced by many factors. Our test examples point out that the performance of the system of Miltenburg, relative to the performance of the CAN^+ system, improves as the ordering cost ratio increases or the potential savings from discounts increase. Apart from cases with very large ordering cost ratios, it depends on the particular situation whether the CAN^+ system outperforms the Miltenburg system or vice versa. Table 5.6 and 5.7 show a 10-item example where the CAN^+ system has a much better performance than the Miltenburg system. However, in general, the cost differences are rather small. (Note that the performance of Miltenburg's system would perhaps improve for the test examples in Table 5.7 if multiple-cycle items are introduced. As mentioned in Section 5.3.1, this aspect is not included in the adapted version of the system of Miltenburg.)

Table 5.6 Data ($A=125$, $L=0.25$, $\beta=0.95$)

i	a_i	h_i	w_i	λ_i	s_i	c_i	S_i
1	80	0.09	0.45	40.63	0	58	328
2	80	7.28	36.41	4.09	0	3	13
3	80	8.43	42.17	34.68	7	20	46
4	80	0.89	4.46	4.24	0	6	35
5	80	0.89	4.46	4.24	0	6	35
6	80	7.28	36.41	4.09	0	3	13
7	80	0.89	4.46	4.24	0	6	35
8	80	0.78	3.92	28.78	0	26	103
9	80	7.28	36.41	4.09	0	3	13
10	80	7.28	36.41	4.09	0	3	13

Note: the data are taken from Silver (1974).

Table 5.7 Relevant cost per time unit (family of Table 5.6)

Q_d	d	CAN+	MIL	MIL-CAN+
1500	0.05	764	1010	246
1500	0.10	649	878	229
1500	0.20	430	704	274
2000	0.05	764	1012	248
2000	0.10	661	882	221
2000	0.20	440	706	276
2500	0.05	790	1020	230
2500	0.10	694	886	192
2500	0.20	474	699	225
3000	0.05	830	1006	176
3000	0.10	769	885	116
3000	0.20	615	710	95
3500	0.05	853	1010	157
3500	0.10	800	900	100
3500	0.20	682	703	21

5.5 Conclusions

This chapter evaluates the performance of the CAN+ system, which has been developed in Chapter 4, by comparing it with the performance of the Miltenburg system. Under this latter inventory control system, three decisions have to be made at a trigger moment: first, a group order quantity for the entire family is selected; then, this order quantity is allocated among the items in the family and, finally, the reorder points are recalculated. Inventory positions of the items are modeled as independent diffusion processes. In order to make a fair numerical comparison between the Miltenburg system and the CAN+ system, we adapted the Miltenburg system for Poisson demands. For this type of demand, the performance of both system has been compared.

The experiments reported in Section 5.4 show that the performance of the CAN+ system is comparable to that of the Miltenburg system as far as the controllable costs are concerned. So, in determining which one to use in practical situations, qualitative arguments will prevail. The following differences between both control systems should be taken into account.

In the first place both systems are primarily developed for different demand processes: a Wiener process for the Miltenburg system and a compound Poisson process for the CAN$^+$ system. Since generalisations to other demand processes are not straightforward for both control systems (except for the simple Poisson demand process), the first conclusion is that for fast movers the Miltenburg system is preferred above the CAN$^+$ system, whereas in case of erratic demand the CAN$^+$ system is preferred.

A second point of consideration is the rate of acceptance of the proposed system by management. In this respect the CAN$^+$ system has some advantages above the Miltenburg system. First, the CAN$^+$ system is a hierarchical system which deals with the economies of scale due to joint ordering costs at the first level (in setting the control parameters), while dealing with economies of scale due to discounts (which usually are quite volatile) only in a second phase, leaving the can-order parameters unchanged. In our opinion a drawback of the Miltenburg system is the resetting of reorder points of the individual items after each order, causing a lot of nervousness in the system.

As far as the required service levels are concerned, we note that in the CAN$^+$ system the realized service is always greater than or equal to the required level, whereas in the Miltenburg system deviations in both directions can occur (although the Miltenburg system gives also higher service levels in most cases).

Finally, we note that in the CAN$^+$ system a discount structure which differs from the all-units discount structure can be taken into account explicitly (see Appendix 4.3), whereas in the Miltenburg system such a different structure can only be dealt with after translation into an all-units discount structure. This provides an additional cause of inaccuracy, because such a translation can only be approximative.

Appendix 5.1 Probability function of the residual stock

In the case of Poisson demand, define T_i, $f_i(t)$ and $F_i(t)$ as in Section 5.3. The probability that item i triggers the following order is equal to the probability that T_i is smaller than all the other T_j. Hence, for i=1,..,N,

$$\Phi_i(0) = Pr\{T_i < T_j, \forall j \neq i\} = \int_{t=0}^{\infty} f_i(t) \prod_{j \neq i} Pr\{T_j > t\}\, dt = \int_{t=0}^{\infty} f_i(t) \prod_{j \neq i} (1 - F_j(t))\, dt\,. \tag{A.5.1}$$

Now, define $T^{(-i)}$ as the time which elapses until any item $j \neq i$ reaches its must-order point if item i is left out of consideration, i.e. $T^{(-i)} = \min_{j \neq i} T_j$, and denote the distribution function and the probability density function of $T^{(-i)}$ by $F^{(-i)}(t)$ and $f^{(-i)}(t)$. So, for i=1,..,N, and $t \geq 0$,

$$F^{(-i)}(t) = 1 - \prod_{j \neq i} (1 - F_j(t))\,,$$

and (A.5.2)

$$f^{(-i)}(t) = \sum_{j \neq i} f_j(t) \prod_{k \neq i,j} (1 - F_k(t))\,.$$

Lemma 5.1

The probability that the residual stock of item i (i=1,..,N) is k is equal to:

$$\Phi_i(k) = \int_{t=0}^{\infty} \frac{(\lambda_i t)^{(\Delta_i - k)}}{(\Delta_i - k)!} e^{-\lambda_i t} f^{(-i)}(t)\, dt, \quad k = 1,..,\Delta_i\,. \tag{A.5.3}$$

Proof

We present a formal proof of (A.5.3) for the case N=2. This proof can be straightforwardly generalized to the case N>2, by replacing T_2 by $\min_{j \neq i} T_j$. Define,

$X_1(t)$: the excess stock above the must-order point of item 1 at time t.

Then, for $k>0$, $\Phi_1(k)=$

$$\int_{t=0}^{\infty} Pr(X_1(\min(T_1,T_2))=k \mid T_1>t, T_2=t)\; dPr(T_1>t, T_2=t)\; +$$

$$\int_{t=0}^{\infty} Pr(X_1(\min(T_1,T_2))=k \mid T_1<t, T_2=t)\; dPr(T_1<t, T_2=t)$$

$$= \int_{t=0}^{\infty} Pr(X_1(t)=k \mid T_1>t, T_2=t)\; dPr(T_1>t, T_2=t)$$

$$= \int_{t=0}^{\infty} Pr(X_1(t)=k \mid T_1>t, T_2=t)\; Pr(T_1>t)\; dPr(T_2=t)$$

$$= \int_{t=0}^{\infty} Pr(X_1(t)=k, T_1>t \mid T_2=t)\; dPr(T_2=t)$$

$$= \int_{t=0}^{\infty} Pr(X_1(t)=k \mid T_2=t)\; dPr(T_2=t)$$

$$= \int_{t=0}^{\infty} \frac{(\lambda_1 t)^{(\Delta_1-k)}}{(\Delta_1-k)!} e^{-\lambda_1 t} f^{(-1)}(t)\, dt\ .$$

Numerical integration can be used to approximate the probability function of the residual stock from (A.5.1) and (A.5.3); see Appendix 5.2.

Another expression for the probability function of the residual stock can be obtained as follows. Let $\gamma_i = \lambda_i/(\Sigma_j \lambda_j)$. Then, $\Phi_i(0)$, $i=1,..,N$, can be calculated from

$$\gamma_i^{\Delta_i} \sum_{j_1=0}^{\Delta_1-1} \gamma_1^{j_1} \cdots \sum_{j_{i-1}=0}^{\Delta_{i-1}-1} \gamma_{i-1}^{j_{i-1}} \sum_{j_{i+1}=0}^{\Delta_{i+1}-1} \gamma_{i+1}^{j_{i+1}} \cdots \sum_{j_N=0}^{\Delta_N-1} \gamma_N^{j_N}\, C_i(j_1,..,j_N)\ , \tag{A.5.4}$$

where,

$$C_i(j_1,..,j_N) = \frac{(\Delta_i - 1 + \sum_{v\neq i} j_v)!}{(\Delta_i-1)! \prod_{v\neq i} (j_v!)}\ . \tag{A.5.5}$$

It can also be shown that $\Phi_i(k)$, for $i=1,..,N$, and $k=1,..,\Delta_i$, is equal to

$$\gamma_i^{\Delta_i-k} \sum_{m \neq i} \gamma_m^{\Delta_m} \sum_{j_1=0}^{\Delta_1-1} \gamma_1^{j_1} \cdots \sum_{j_{i-1}=0}^{\Delta_{i-1}-1} \gamma_{i-1}^{j_{i-1}} \sum_{j_{i+1}=0}^{\Delta_{i+1}-1} \gamma_{i+1}^{j_{i+1}} \cdots \sum_{j_{m-1}=0}^{\Delta_{m-1}-1} \gamma_{m-1}^{j_{m-1}} \sum_{j_{m+1}=0}^{\Delta_{m+1}-1} \gamma_{m+1}^{j_{m+1}} \cdots \sum_{j_N=0}^{\Delta_N-1} \gamma_N^{j_N} C_{i,m}(j_1,..,j_N) \,, \tag{A.5.6}$$

where,

$$C_{i,m}(j_1,..,j_N) = \frac{(\Delta_i - k + \Delta_m - 1 + \sum_{v \neq i,m} j_v)!}{(\Delta_i - k)!\,(\Delta_m - 1)! \prod_{v \neq i,m} (j_v!)} \,. \tag{A.5.7}$$

It is obvious that this expression is numerically intractable when N or Δ_i (for some i) is large.

Appendix 5.2 Numerical integration

In this appendix we present a standard numerical integration method which has been used to solve the integral equations (A.5.1) and (A.5.3). Note that these equations have the following form: $\int_{t=0}^{\infty} \mathscr{L}(t)\,dt$, where $\mathscr{L}(t)$ is a function of t which can not be simplified.

Algorithm for numerical integration

Step 1: Determine t_{max} such that the cumulative probability density of t higher than t_{max} is less than 10^{-3}.

Step 2: Determine 25 integration points, where $u_1 := 0$ and $u_j := u_{j-1} + t_{max}/24$ for $j=2,..,25$.

Step 3: The function $\mathscr{L}(t)$ is approximated by a piece-wise linear function

$$\hat{\mathscr{L}}(t) = a_j t + b_j \quad for\ t \in [u_j, u_{j+1}] \quad j=1,..,24 \,, \tag{A.5.8}$$

where a_j and b_j are given by

$$a_j = \frac{\mathscr{L}(u_{j+1}) - \mathscr{L}(u_j)}{u_{j+1} - u_j} \,, \quad b_j = \frac{\mathscr{L}(u_j)u_{j+1} - \mathscr{L}(u_{j+1})u_j}{u_{j+1} - u_j} \,. \tag{A.5.9}$$

Calculate a_j and b_j for $j=1,..,24$.

Step 4: Calculate

$$\int_{t=0}^{\infty} \mathcal{L}(t)\,dt \approx \sum_{j=1}^{24} \int_{t=u_j}^{u_{j+1}} \hat{\mathcal{L}}(t)\,dt = \sum_{j=1}^{24} \int_{t=u_j}^{u_{j+1}} \{a_j t + b_j\}\,dt = \tag{A.5.10}$$

$$\sum_{j=1}^{24} \left\{ \frac{1}{2} a_j (u_{j+1}^2 - u_j^2) + b_j (u_{j+1} - u_j) \right\} .$$

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CHAPTER 6

ON THE DETERMINATION OF THE CONTROL PARAMETERS OF THE OPTIMAL CAN-ORDER POLICY

This chapter considers the determination of the optimal can-order policy. Recent comparative studies have pointed out that the performance of the optimal can-order policy is poor, compared with other coordinated replenishment strategies, when the ordering cost ratio is large. It will be shown that it is the *method* to calculate the optimal can-order parameters which causes the bad performance in such situations and not the policy itself. Attention is focused to a subclass of can-order policies, which is close to the optimal can-order policy for large ordering cost ratios. A solution procedure is developed to calculate the optimal control parameters of this policy. It is shown that a properly chosen combination of the solution procedures to calculate can-order parameters leads to a can-order strategy which performs as good as other coordinated replenishment policies.

6.1 Introduction

The CAN$^+$ system which has been proposed in Chapter 4, uses the can-order policy as a basic strategy. Can-order policies, which are characterized by a set of three parameters (S_i, c_i, s_i) for each item i, focus on reducing joint ordering costs by coordinated replenishments of several items: if a replenishment is triggered, because the inventory position of an item i falls to or below the must-order point s_i, then any item j with an inventory position at or below its can-order point c_j is included in the joint replenishment. The inventory position of every item j in the order is raised up to its order-up-to level S_j. In this chapter the determination of the parameters of the optimal can-order policy will be investigated.

Several other inventory control policies have been proposed for the coordinated

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replenishment problem with a joint ordering cost structure. (An overview has been given in Chapter 1.) Recently, Atkins and Iyogun (1988) and Pantumsinchai (1992) compared the performance of different coordinated replenishment policies under Poisson demands. They concluded from their empirical results that the optimal can-order strategy is outperformed quite frequently by other coordinated replenishment strategies. The performance of the can-order policy becomes poor as the joint ordering cost (relative to the average individual ordering cost) increases.

In these comparative studies, the can-order parameters were calculated by the method of Federgruen et al. (1984). (This method has also been used to determine the parameters of the basic ordering strategy in the CAN^+ system.) This heuristic is based on a decomposition of the multi-item problem into a number of single-item problems. It will be shown that the bad performance of the can-order policy is due to the decomposition assumption which is used by Federgruen et al. As a consequence, it is the *method* to calculate the can-order parameters which performs bad in situations with large ordering cost ratios, but not the can-order *policy* itself. For large ordering cost ratios (i.e. the ratio of the joint ordering cost and the average individual ordering cost), attention is restricted to the subclass of can-order policies with $c_i = S_i - 1$ for all items i. Under this (S,S-1,s) policy all items are jointly reordered as soon as one item reaches its must-order point.

Section 6.2 discusses the approximate decomposition method to determine the optimal can-order parameters. In Section 6.3 we analyze the (S,S-1,s) policy, which is close to the optimal can-order policy for large ordering cost ratios, and we develop a solution procedure to determine optimal parameters of this policy. Section 6.4 compares the performance of the (S,S-1,s) policy with the performance of the can-order policy, obtained by the approximate decomposition method, as well as other coordinated replenishment policies. The major conclusions are summarized in Section 6.5.

6.2 Analysis of the approximate decomposition method

For the sake of clearness, the model assumptions are repeated first. We consider a family of N items with demands generated by independent Poisson processes with rate λ_i for item i. Shortages are completely backlogged. The replenishment lead time of an order is deterministic and equals L periods. There is a joint ordering cost, A, associated with any order, and an individual ordering cost, a_i, for each item i included in the replenishment. Let $\bar{a}$ be the average individual ordering cost, then the ordering cost ratio is defined by

$A/\bar{a}$. Holding costs are charged at a rate h_i per period on every unit of item i on stock. The management requires that a given fraction β of demand has to be satisfied directly from stock on hand. As opposed to the situation in Chapter 4 and 5, there are no discounts available. The objective is to minimize the sum of the long run average holding and ordering costs subject to the service constraint.

The determination of the optimal control parameters of the can-order policy is complicated by the interaction between items. When an order is triggered by item i, because its inventory position falls to s_i, this represents a *special replenishment opportunity* to order at reduced ordering costs for all the other items. Silver (1974) suggested to decompose the N-item problem in N single-item problems by assuming that special replenishment opportunities for item j occur according to a Poisson process with rate μ_j, which is equal to the sum of the expected number of replenishments per unit time, ξ_i, of the other items $i \neq j$. This idea was used in the papers by Silver (1974, 1981), Thompstone and Silver (1975), and Federgruen et al. (1984). Federgruen et al. used a specialized policy iteration algorithm to calculate the optimal parameters S_i, c_i, s_i in the resulting single-item problem for item i with special replenishment opportunities occurring at a given rate μ_i. The actual rates μ_i of special replenishment opportunities are calculated by an iterative procedure.

Approximate decomposition method for computing optimal can-order parameters

Step 0: Choose starting values for ξ_i $(i=1,..,N)$.

Step 1: Iteration step:

a) Initialize $i:=0$.

b) Set $i:=i+1$, compute $\mu_i := \Sigma_{j \neq i} \xi_j$.

c) Solve the single-item problem; i.e. choose the set (S_i, c_i, s_i) which minimizes the expected long-run average cost of item i per unit time, subject to a given service level constraint, when demands and special replenishment opportunities are generated by independent Poisson processes with rates λ_i and μ_i, respectively.

d) Compute ξ_i given the parameters (S_i, c_i, s_i).

e) Go to Step 2 if $i=N$; otherwise go to Step 1b.

Step 2: Termination:

Stop when the new control parameters of all items are the same as in the previous iteration; otherwise go to Step 1a.

Silver (1974) already noted that this *special replenishment opportunity model* tends to overestimate the real cost and to underestimate the real service. The finding with respect to the overestimation of the real cost is confirmed by the simulation results in Table 6.1. This table shows the percentage overestimation of the cost by the model for a family consisting of ten identical products with the following parameters: L=1, β=0.95, a_i =1, h_i =0.25, λ_i =5, i=1,..,10. It turns out that the percentage cost error may be considerable. The extent of overestimation of the real cost increases as the ordering cost ratio increases.

Table 6.1 Comparison of model cost and simulated cost

$A/\bar{a}$	model cost	simulated cost	% cost error
0	22.57	22.57	0.00
1	24.45	24.35	0.41
2	25.96	25.53	1.68
5	30.51	28.86	5.72
10	36.97	32.76	12.85
20	49.49	41.58	19.02
50	64.22	48.45	32.55

Note: % cost error=100 · (model cost-simulated cost)/simulated cost

The conclusions in the comparative studies of Atkins and Iyogun (1988) and Pantumsinchai (1992) are based on the cost which is computed from the *model* of Federgruen et al. As can be seen in Table 6.1, the model cost differs significantly from the real cost in some situations. Therefore, it would have been better to use the real (simulated) cost to measure the performance of the can-order strategy.

When the ordering cost ratio is zero, then the optimal can-order policy will be an independent policy with $c_i = s_i$ for all items i. On the other hand, when the ordering cost ratio is infinite (because the individual ordering cost is negligible for each item), then the optimal policy has $c_i = S_i - 1$ for all items, which implies that all items are jointly replenished as soon as an item triggers an order. (Since $c_i = S_i - 1$, an item is not ordered if there has been no demand for it after the preceding order.) The above mentioned two special policies can be considered as extreme policies within the class of possible can-order policies.

One may imagine that the optimal can-order policy will tend to a (S,S-1,s) policy for large ordering cost ratios. Since all items are ordered simultaneously under a (S,S-1,s) policy, the control parameters (S_i ,s_i , i=1,..,N) have to be chosen such that the residual stock (i.e. the stock above the must-order point when an order is triggered) will be close to zero for every item. This implies that during a cycle between two trigger moments the probability of a special replenishment opportunity will be rather small in the beginning of the cycle and large at the end. This contradicts with the approximate assumption of Poisson arrivals of special replenishment opportunities, which is made by Silver, Federgruen, and others. Numerical examples point out that the overestimation of the cost of a reasonable (S,S-1,s) strategy is very large if the method of Federgruen or Silver is used. In fact, their models will hardly suggest a strategy of (S,S-1,s)-type because the cost of such a strategy is overestimated even more than can-order strategies with other parameter settings. (See also Section 6.4.)

Hence, we conclude that the approximate decomposition method to determine the can-order parameters leads to bad results for large ordering cost ratios because in this situation the optimal solution does not satisfy the assumption of Poisson arrivals of special replenishment opportunities. In the next section, an alternative solution method is proposed for these cases. This method determines the parameters of a (S,S-1,s) policy, which is, in general, close to the optimal can-order policy in situations with large ordering cost ratios.

6.3 Determination of the parameters of the optimal (S,S-1,s) policy

This section is divided in three parts. In the first part, a cost expression is derived for a given (S,S-1,s) strategy. In the second part, a method is developed to determine the optimal must-order point s_i (i=1,..,N) given a vector $\Delta:=(\Delta_1,..,\Delta_N)=(S_1-s_1 ,..,S_N -s_N)$. Finally, the results of the first and the second part are used in the third part, which presents a heuristic to determine the optimal parameters of a (S,S-1,s) policy.

6.3.1 Cost expression for a given (S,S-1,s) strategy

Note that the inventory position of each item i equals S_i at the beginning of an order cycle, which ends as soon as any item reaches its must-order point. The stochastic process, which describes the changes in the vector of the inventory positions just before an order, is a discrete-time Markov chain with a finite state space.

For a given (S,S-1,s) strategy, define:

C : long run average cost per unit time;

p_i^0 : probability that no demand arrives for item i during an order cycle (i=1,..,N);

η_i : expected holding cost of item i during an order cycle (i=1,..,N);

τ : expected length of an order cycle.

Then, from the theory of regenerative processes, it follows that

$$C = \frac{A + \sum_{i=1}^{N} \left\{ (1 - p_i^0) a_i + \eta_i \right\}}{\tau} . \tag{6.1}$$

Suppose an order cycle starts at time 0. To analyze the expected (order) cycle time, define the following stochastic variables:

T_i : time until the cumulative demand for item i reaches the level $\Delta_i := S_i - s_i$ (i=1,..,N);

T : time until *any* item triggers an order.

Note that item i will trigger an order as soon as the total demand for item i from time 0 onwards equals Δ_i . Because demands for individual items are generated according to independent Poisson processes, it follows that T_i is Erlang distributed with parameters λ_i and Δ_i . Denote the corresponding probability density function and the distribution function by $f_i(t)$ and $F_i(t)$, respectively. Noting that $T = \min_i T_i$ it follows that the distribution function and the density function of T, denoted by F(t) and f(t) respectively, are given by

$$F(t) = 1 - \prod_{i=1}^{N} (1 - F_i(t)) , \tag{6.2}$$

and,

$$f(t) = \sum_{i=1}^{N} f_i(t) \prod_{j \neq i}^{N} (1 - F_j(t)) . \tag{6.3}$$

The expected length of an order cycle is then given by

$$\tau = \int_{t=0}^{\infty} \{1 - F(t)\} dt = \int_{t=0}^{\infty} \{\prod_{i=1}^{N} (1 - F_i(t))\} dt . \tag{6.4}$$

This integral can be approximated arbitrarily close by numerical integration.

Define:

$\Phi_i(k)$: the probability that at time T the residual stock of item i equals k;

$H_i(x,y,t)$: expected total holding cost for item i during an order cycle of t periods given that the inventory on hand equals x at the beginning and equals y at the end of the cycle.

The probability mass function, $\Phi_i(k)$, of the residual stock of item i ($i=1,..,N$), is presented in Appendix 6.1. (It appears that this probability function is similar to the one which has been derived for the residual stock in the Miltenburg system; see Chapter 5.)

Consider the expected holding cost per order cycle in case the lead time is negligible. Then, the inventory on hand of item i decreases from S_i to s_i+k ($k=0,..,\Delta_i$) with probability $\Phi_i(k)$ during an order cycle. If the order cycle time is t periods, the expected holding cost during that cycle equals $H_i(S_i, s_i+k, t)$. A general expression for $H_i(x,y,t)$ is derived in Appendix 6.2.

For a positive lead time L, the holding cost in [L,T+L] is shifted back to the interval [0,T]. Because the demand for item i during the lead time L is generated by an independent Poisson process with rate $\lambda_i L$, the inventory on hand at time L equals $S_i - j$ with probability $\frac{(\lambda_i L)^j}{j!} e^{-\lambda_i L}$. Now, it is easily seen that

$$\eta_i = \sum_{j=0}^{\infty} \frac{(\lambda_i L)^j}{j!} e^{-\lambda_i L} \sum_{k=0}^{\Delta_i} \Phi_i(k) \int_{t=0}^{\infty} H_i(S_i - j, s_i + k - j, t) f(t)\, dt \,. \tag{6.5}$$

Using formula (A.6.1), (A.6.2), and (A.6.4), equation (6.5) can be approximated arbitrarily close by numerical integration.

Finally, the probability p_i^0 is equal to $\Phi_i(S_i - s_i)$. This completes the derivation of the elements of cost formula (6.1).

6.3.2 Determination of the must-order points

This subsection investigates the determination of the must order points given a vector $\Delta=(\Delta_1,..,\Delta_N)=(S_1-s_1,..,S_N-s_N)$. The problem is to find the smallest value of s_i ($i=1,..,N$) such that a given fraction of demand, β, is satisfied directly from stock on hand.

Define, for a given (S,S-1,s) strategy, for item i:

β_i : long run fraction of demand satisfied directly from stock on hand (i=1,..,N);

ES_i : expected number of shortages during an order cycle (i=1,..,N);

EQ_i : expected order quantity per order cycle (i=1,..,N).

From the theory of regenerative processes, it follows that

$$\beta_i = 1 - \frac{ES_i}{EQ_i} . \tag{6.6}$$

Recall that Φ_i (k) is the probability of having a residual stock of k units for item i at time T and that the demand for item i during the lead time is generated by a Poisson process with rate $\lambda_i L$. Then it easily follows that

$$ES_i = \sum_{k=0}^{\Delta_i} \Phi_i(k) \sum_{j=s_i+k}^{\infty} (j-s_i-k) \frac{(\lambda_i L)^j}{j!} e^{-\lambda_i L} - \sum_{j=s_i+\Delta_i}^{\infty} (j-s_i-\Delta_i) \frac{(\lambda_i L)^j}{j!} e^{-\lambda_i L} . \tag{6.7}$$

Furthermore,

$$EQ_i = \sum_{k=0}^{\Delta_i} (\Delta_i - k) \Phi_i(k) . \tag{6.8}$$

Once the probability function Φ_i (k) of the residual stock has been calculated, β_i can be obtained from (6.6), (6.7), and (6.8).

Algorithm to determine s_i given the vector Δ

Step 1: Determine the probability function Φ_i (k), k=0,..,Δ_i, from (A.6.1) and (A.6.2).

Step 2: a) Initialize $s_i := 0$; calculate EQ_i from (6.8).

b) Calculate ES_i from (6.7).

c) Stop if $ES_i < (1-\beta)EQ_i$; otherwise increase s_i by one unit and go back to Step 2b.

6.3.3 Determination of the optimal vector Δ

The results of Section 6.3.1 and 6.3.2 can be used to determine the optimal must-order points and the corresponding cost for a *given* vector Δ. Now, an iterative solution method will be proposed to find an approximation for the vector Δ of the optimal (S,S-1,s) policy. The heuristic is outlined in the following algorithm.

Algorithm to determine the optimal vector Δ

Step 1: Determine $$T_D := \sqrt{\frac{2(A+\sum_{i=1}^{N} a_i)}{\sum_{i=1}^{N} \lambda_i h_i}}\,; \tag{6.9}$$

For all items i, determine the integer value of Δ_i for which the difference between $\sum_{j=0}^{\Delta_i} \frac{(\lambda_i T_D)^j}{j!} e^{-\lambda_i T_D}$ *and* $\frac{N}{N+1}$ is minimal;

Determine the corresponding must-order points by the method of Section 6.3.2 and calculate the cost C by formula (6.1); Set C_{min}=C.

Step 2: Set i:=0;

Repeat (until i=N)

- i:=i+1;
- Carry out a one dimensional search on Δ_i by the Golden-Section method;
- Update Δ_i and C_{min} if a better solution has been found.

Step 3: Stop if the vector Δ has not been changed in Step 2 or C_{min} has not been decreased by more than ϵ%; otherwise return to Step 2.

The starting value for Δ (Step 1) has been suggested by Love (1979) in a related context (Love provides no motivation for this heuristic). Note that T_D is the optimal length of an order cycle in the deterministic demand case. (From prior numerical examples it appeared that the obvious choice of $\Delta_i = \lambda_i T_D$ does not work satisfactorily.)

In every iteration (Step 2), a one dimensional search is carried out for each item: Δ_i is varied, while the other Δ_j, $j \neq i$, remain the same. To save computation time, the Golden-Section method is used (which assumes convexity of C in Δ_i). For every evaluation of a possible value of Δ_i, the must order points of *all* items have to be calculated (since Δ_i can also affect other must-order points), together with the corresponding cost for the *whole* family. The iterative process terminates as soon as the vector Δ remains the same in two successive iterations or the minimal cost has been decreased less than a specified percentage of ϵ%.

6.4 Numerical results

The above procedure has been applied on several numerical examples. Two families of items are considered, consisting of 4 and 8 items. The values of λ_i, a_i, and h_i are listed in Table 6.2 for both families. For different experiments the lead time L is varied over two levels (0.2 and 1), the required service level β is also varied over two levels (0.95 and 0.99), and the joint ordering cost A is varied over three levels (25, 250, 500). Detailed results of the 24 examples are given in Appendix 6.3.

Table 6.2 Data for numerical examples

family with N=4				family with N=8			
item i	λ_i	a_i	h_i	item i	λ_i	a_i	h_i
1	20	10	5	1	20	10	5
2	15	20	5	2	15	10	5
3	10	30	5	3	10	20	5
4	5	40	5	4	5	20	5
				5	20	30	5
				6	15	30	5
				7	10	40	5
				8	5	40	5

The performance of a given coordinated replenishment strategy is measured by the percentage cost saving over the optimal independent (S,s) strategy. The optimal (S,s) strategy can be obtained by the approach of Federgruen et al. (1984) with $\mu_i = 0$ and $c_i = s_i$ for all items i. The cost which is computed from the model is exact because no assumption on the arrival process of the special replenishment opportunities is needed ($\mu_i = 0$). The percentage cost saving is calculated as

$$\% c.s. = 100 \cdot \frac{cost\ of\ (S,s)\ strategy - cost\ of\ coordinated\ strategy}{cost\ of\ (S,s)\ strategy} . \tag{6.10}$$

First, the performance of the optimal (S,S-1,s) strategy is compared with the performance of the optimal (S,c,s) strategy, obtained by the approximate decomposition method of Federgruen et al. Table 6.3 gives the average percentage cost saving for fixed values of the ordering cost ratio A/ā. Note that the performance of the (S,c,s) strategy is based on the real (simulated) cost of the strategy that follows from the model.

Table 6.3 Average % c.s. of optimal (S,S-1,s) and (S,c,s) policy

A/ā	(S,S-1,s)	(S,c,s)
1	0.07	8.88
10	34.80	30.73
20	41.02	35.43

Note: the average performance for a fixed ordering cost ratio is based on 8 observations.

As expected, the (S,S-1,s) policy performs less than the (S,c,s) policy for the small ordering cost ratio. In some individual cases, the optimal (S,S-1,s) strategy has even a higher cost than the optimal (S,s) strategy. However, the (S,S-1,s) policy outperforms the (S,c,s) policy for large ordering cost ratios. It can be noted that the differences would even be larger if the cost from the *model* of Federgruen et al. had been used, as Atkins and Iyogun (1988) and Pantumsinchai (1992) do.

Atkins and Iyogun (1988) and Pantumsinchai (1992) conclude from their numerical experiments that the can-order policy may perform very poor, relative to other coordinated replenishment policies, for large ordering cost ratios. In these situations, we recommend to use the (S,S-1,s) policy, where the parameters are determined by the method in Section 6.3. The model of Federgruen et al. should be used for small ordering cost ratios. Let the CAN policy be the best can-order strategy in a given situation. Based on numerical experience, we suggest the following rule of thumb to determine the parameters of the CAN policy:

Procedure to determine the parameters of the CAN policy

- If A/ā≤2 : Use the model of Federgruen et al. (1984) to determine the parameters S_i, c_i, and s_i for each item i.
- If 2<A/ā<5 : Determine the parameters S_i, c_i, and s_i for each item i with the model of Federgruen et al. and with the model in Section 6.3. Choose the parameters according to the strategy with the lowest cost.

- If $A/\bar{a} \geq 5$: Use the method of Section 6.3 to obtain the parameters Δ_i and s_i for all items i. Set $S_i := s_i + \Delta_i$ and $c_i := S_i - 1$.

The conclusions in Pantumsinchai (1992) and Atkins and Iyogun (1988) are based on a comparison of the optimal can-order policy with the optimal (Q,S) and (R,S) policy. Under a (Q,S) policy, the inventory position of all items j is raised up to the order-up-to level S_j whenever the combined inventory position of all the items drops to or below the group reorder point. Under unit demand sizes, the group reorder point is reached whenever the total demand since the last order reaches Q. The (R,S) policy is a periodic review policy, determined by the parameters (R_i, S_i) for every item i, where the inventory position of item i is ordered up to S_i every R_i periods. To achieve coordination, the review intervals R_i are chosen as multiples k_i of some basic period.

Atkins and Iyogun (1988) and Pantumsinchai (1992) give algorithms to calculate the optimal parameters for respectively a (R,S) policy and a (Q,S) policy in case of stock-out costs and Poisson demands. However, it is easy to adapt their algorithms to the service level case. In the numerical experiments we use a (R,S) policy with an equal review period for each item ($R_i = R$ for all i).

The average performance of CAN and the optimal (Q,S) and (R,S) strategy is shown for fixed values of $A/\bar{a}$, L, and β in Table 6.4. Note that the numbers in Table 6.4 are averages of the numbers in the detailed list in Table A.6.1. To compare our results with the results of the other comparative studies, the performance according to the cost from the *model* of Federgruen et al. (FED) has also been calculated.

By comparing the performance of the (R,S) and the (Q,S) policy on one side and the FED policy on the other side, the same conclusions can be drawn as in earlier studies. However, if the performance of the (R,S) and the (Q,S) policy is compared with that of the CAN policy, then it appears that the can-order policy performs at least equally well as the other policies, even for large ordering cost ratios. This supports our conjecture that the poor performance of the can-order policy in some cases is due to the method to determine the control parameters and not to the policy itself.

It seems that the service level has a significant impact on the percentage cost saving of, in particular, the (R,S) and the (Q,S) policy. (This was already noted by Pantumsinchai (1992) for the (Q,S) policy with respect to the stock-out cost.) The percentage cost savings are higher for the family of 8 items. However, the relative performance of the different policies is not affected by the number of items.

Table 6.4 Average % c.s. of several coordinated replenishment policies

factor	RS	QS	CAN	FED
$A/\bar{a}$ (8 observations)				
1	-1.89	0.93	8.88	7.72
10	34.70	35.71	34.78	21.83
20	40.83	42.13	41.01	24.26
L (12 observations)				
0.2	24.29	26.44	28.78	18.25
1.0	24.39	26.07	27.69	17.63
β (12 observations)				
0.95	28.41	29.92	30.19	19.34
0.99	20.27	22.58	26.28	16.53

This section will be closed with some remarks on the misspecification in the cost when the special replenishment opportunity model is used. It has already been mentioned that the percentage cost error will be very large for an arbitrary (S,S-1,s) strategy. This conjecture is verified by calculating the cost of the optimal (S,S-1,s) strategy (obtained with the approach in Section 6.3) with the method of Federgruen et al. Recall that $\mu_i := \Sigma_{j \neq i} \xi_j$, where ξ_i denotes the expected number of replenishments per unit time that is triggered by item i. Note that ξ_i is equal to $\Phi_i(0)/\tau$ in the (S,S-1,s) model. The average percentage cost error of the (S,c,s) strategy, calculated by the approach of Federgruen et al., has also been calculated. Table 6.5 shows that the average percentage cost error is very high for large ordering cost ratios. The cost errors are dramatic for the (S,S-1,s) strategy. Hence, the model of Federgruen et al. will neglect such a policy, when searching for the optimal can-order policy.

Table 6.5 Average percentage cost error of (S,c,s) and (S,S-1,s) strategy

$A/\bar{a}$	(S,c,s)	(S,S-1,s)
1	1.23	5.75
10	11.85	30.18
20	14.91	37.73

Note: % cost error = 100 · (model cost-simulated cost)/simulated cost

6.5 Conclusions

Our analysis shows that can-order policies indeed do not outperform other coordinated replenishment policies such as (R,S) or (Q,S) policies. Nevertheless, the conclusions made in the comparative studies of Atkins and Iyogun (1988) and Pantumsinchai (1992) are wrong. It has been shown that the performance of the can-order policy ought not to be evaluated by the special replenishment opportunity model, suggested by Silver (1974) and Federgruen et al. (1984), in situations with large ordering cost ratios, because this model gives inaccurate results in such circumstances. For the case of Poisson demands, we developed a solution method to find the parameters of a (S,S-1,s) policy, which is a close to optimal can-order policy in situations with large ordering cost ratios. Numerical analysis points out that a properly chosen combination of both solution techniques leads to a can-order strategy which performs as well as the optimal (R,S) or (Q,S) policy, as distinct from conclusions in the above mentioned comparative studies.

Appendix 6.1 Probability function of the residual stock

It turns out that the problem of determining the probability function of the residual stock is an important issue. In Chapter 5 we solved exactly the same problem when adapting the Miltenburg system for Poisson demands. For the sake of completeness, the formulas are repeated below. For more details on the derivation of formula (A.6.1) up to (A.6.3), we refer to Appendix 5.1.

Define $f_i(t)$ and $F_i(t)$ as in Section 6.3. Then, for $i=1,..,N$,

$$\Phi_i(0) = \int_{t=0}^{\infty} f_i(t) \prod_{j \neq i} (1 - F_j(t))\, dt\ , \tag{A.6.1}$$

and,

$$\Phi_i(k) = \int_{t=0}^{\infty} \frac{(\lambda_i t)^{(\Delta_i - k)}}{(\Delta_i - k)!} e^{-\lambda_i t} f^{(-i)}(t)\, dt, \quad k=1,..,\Delta_i, \tag{A.6.2}$$

where,

$$f^{(-i)}(t) = \sum_{j \neq i} f_j(t) \prod_{k \neq i,j} (1 - F_k(t))\ ,\ t \geq 0\ . \tag{A.6.3}$$

Appendix 6.2 Determination of $H_i(x,y,t)$

Recall that $H_i(x,y,t)$ denotes the expected holding cost for item i during an order cycle of t periods given that the inventory on hand equals x at the beginning and equals y at the end of the cycle. It can be shown that the (x-y) demands are homogeneously distributed over [0,t] (see e.g. Tijms (1986)). Five different situations are distinguished, depending on whether x and y are positive or negative and whether the particular item triggers the order or not. Note that in case $x-y=\Delta_i$ the last demand of item i was at time t (since the item triggers the order). The following formula for $H_i(x,y,t)$ summarizes all five different cases:

$$H_i(x,y,t) = \begin{cases} h_i \dfrac{t}{2}(x+y) & if\ x>0, y\geq 0, x-y<\Delta_i, \\ h_i \dfrac{t}{2}(x+y+1) & if\ x>0, y\geq 0, x-y=\Delta_i, \\ h_i \dfrac{t}{2}\dfrac{x(x+1)}{(x-y+1)} & if\ x>0, y<0, x-y<\Delta_i, \\ h_i \dfrac{t}{2}\dfrac{x(x+1)}{(x-y)} & if\ x>0, y<0, x-y=\Delta_i, \\ 0 & if\ x\leq 0\ . \end{cases} \tag{A.6.4}$$

Appendix 6.3 Extended numerical results

The integral equations (6.3), (6.5), (A.6.1) and (A.6.2) have been solved by the method as described in Appendix 5.2.

The detailed results for each example are presented in Table A.6.1. The input-parameters N, L, β, and A are already defined. The variables C1 up to C7 are explained below.

Legend to Table A.6.1.

C1: cost calculated from the model of Federgruen et al. for the (S,c,s) strategy obtained by the same model;

C2: simulated cost for the (S,c,s) strategy obtained by the model of Federgruen et al.;

C3: cost calculated from the model of Federgruen et al. for the (S,S-1,s) strategy obtained by the algorithm in Section 6.3;

C4: exact cost calculated by formula (6.1) for the (S,S-1,s) strategy obtained by the algorithm in Section 6.3;

C5: exact cost according to optimal (R,S) policy obtained by an adapted version of the method of Atkins and Iyogun (1988) (the optimal R was found by a grid search with steps of 0.05);

C6: exact cost according to the optimal (Q,S) policy obtained by an adapted version of the method of Pantumsinchai (1992);

C7: exact cost according to the optimal (S,s) policy obtained by the model of Federgruen et al. (1984).

Table A.6.1 Detailed numerical results

ex.	N	L	β	A	C1	C2	C3	C4	C5	C6	C7
1	4	0.2	0.95	25	279.7	275.2	321.7	299.1	307.4	294.4	308.8
2	4	0.2	0.95	250	563.2	493.0	626.5	469.0	467.9	456.7	693.7
3	4	0.2	0.95	500	766.2	655.2	864.2	608.3	590.3	577.5	955.0
4	4	0.2	0.99	25	316.1	311.5	365.4	342.7	370.3	357.4	333.4
5	4	0.2	0.99	250	606.4	541.9	670.5	520.0	556.1	531.2	737.5
6	4	0.2	0.99	500	820.9	713.1	923.6	658.5	693.6	667.7	1003.7
7	4	1.0	0.95	25	314.5	311.6	349.5	335.3	333.0	326.1	341.8
8	4	1.0	0.95	250	579.5	521.5	638.0	484.1	488.7	477.2	711.9
9	4	1.0	0.95	500	761.4	666.9	799.7	606.0	608.0	598.9	972.6
10	4	1.0	0.99	25	374.4	371.0	418.4	400.3	420.0	402.5	385.8
11	4	1.0	0.99	250	659.5	598.7	719.2	569.0	588.7	571.4	788.6
12	4	1.0	0.99	500	847.8	747.5	957.2	700.5	721.8	700.7	1044.4
13	8	0.2	0.95	25	547.9	538.4	*	599.5	592.3	576.8	622.0
14	8	0.2	0.95	250	1007.2	866.6	*	812.3	771.7	764.6	1391.3
15	8	0.2	0.95	500	1356.7	1111.3	*	1004.6	935.3	920.3	1912.9
16	8	0.2	0.99	25	625.1	616.1	*	696.5	711.6	697.7	671.5
17	8	0.2	0.99	250	1103.9	965.9	*	913.5	835.3	911.2	1479.1
18	8	0.2	0.99	500	1429.2	1186.2	*	1078.8	1112.3	1085.3	2010.5
19	8	1.0	0.95	25	617.9	611.4	*	652.4	643.3	636.1	685.8
20	8	1.0	0.95	250	1049.1	924.9	*	830.7	827.4	814.6	1427.8
21	8	1.0	0.95	500	1344.6	1117.8	*	992.0	977.5	965.4	1943.6
22	8	1.0	0.99	25	730.1	722.9	*	810.7	814.7	796.8	789.9
23	8	1.0	0.99	250	1197.7	1066.3	*	1018.8	1011.7	993.8	1566.8
24	8	1.0	0.99	500	1532.5	1299.8	*	1170.1	1177.5	1159.9	2092.0

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CHAPTER 7

MULTI-ITEM INVENTORY SYSTEMS WITH JOINT ORDERING AND TRANSPORTATION DECISIONS

In many practical situations joint determination of ordering and transportation decisions for a family of items may lead to a considerable cost saving. In this chapter we consider a multi-item inventory system with a freight rate schedule that is a function of the volume shipped and the capacity of a standard container. Orders for all items of the family are triggered by a coordinated periodic (R,S) policy. Economies of scale exist because of reduced freight rates when ordering a full-container load instead of a less-than-container load. A full-container load can be achieved by enlarging the initial order quantities. A heuristic is proposed to decide whether an initial order should be enlarged or not. Some numerical examples show that the heuristic works quite satisfactorily.

7.1 Introduction

In many practical situations inventory control and transportation planning are closely related. However, the incorporation of transportation cost into the analysis of order quantities has received scant attention in the literature. Usually it is assumed that the freight rate is constant, if considered at all. In practice, the transportation cost structure reflects considerable reductions in freight rates when the shipped quantities exceed some nominal rate breakpoints. Freight rate discount schedules are quite similar to quantity discount schedules, such as all-units or incremental-units discount schedules, but are usually based on weight, volume, carload-lot or standard container sizes, instead of units. Recently, Tersine and Barman (1991) and Russell and Krajewski (1991) presented methods, that incorporate both quantity and freight discounts into the order sizing decisions in a deterministic single-item inventory system. Jucker and Rosenblatt (1985)

This chapter is based on a paper with the same title, which has been accepted for publication in *International Journal of Production Economics*.

and Pantumsinchai and Knowles (1991) considered freight rate discounts in the context of a single-period inventory model. They provided order sizing algorithms for combined quantity and freight rate discount schedules. Anily and Federgruen (1993) and Diaby and Martel (1993) proposed methods to determine optimal purchasing and shipping quantities in multi-echelon distribution systems with deterministic demand.

All these papers consider single-item problems without coordination of orders of different items. There are several reasons for coordinating items when making replenishment decisions for items which are stocked at the same location and have the same supplier. In the literature most coordinated replenishment systems focus on reducing joint ordering costs. (See Chapter 1, 2, 3, and 6 for relevant references.) Other reasons for coordinated control include quantity or freight rate discounts. In many cases it may be uneconomical to achieve a discount breakpoint quantity by ordering one single item, but it could certainly make sense to coordinate several items to achieve such a discount breakpoint or container load. In Chapter 4 and 5 we investigated classes of coordinated replenishment strategies which, among other factors, account for quantity discount schedules in a stochastic environment.

In this chapter we consider a multi-item inventory system with a freight rate schedule that is a function of the volume shipped and the capacity of a standard container. Economies of scale exist because of reduced shipping rates when ordering a full-container load instead of a less-than-container load. To solve this stochastic multi-item problem, a procedure is presented for the determination of the order composition, which accounts for both inventory and transportation related costs.

In Section 7.2 a description of the problem is given together with a periodic ordering strategy, that allows to enlarge initial order quantities to achieve economies of scale. In Section 7.3 approximations are derived for the expected saved ordering cost, the expected extra holding cost, and the expected saved shipping cost, caused by such an enlargement. We propose a heuristic to calculate the composition of the order. Some numerical results are presented in Section 7.4. Finally, the conclusions are given in Section 7.5.

7.2 Description of the problem

We consider a family of N items which are stocked at a single central warehouse. The family of items is ordered from a single supplier overseas. Inventories are periodically reviewed. At every review period the central warehouse may place an order for one or more of the items. This order arrives L time units later (the lead time L is constant).

Demands for item i in subsequent periods are independent identically distributed random variables with a general distribution function with expectation μ_i and variance σ_i^2. The demand processes for the various items are supposed to be independent of each other. Excess demands are backlogged.

The objective is to minimize the total long run expected cost per unit time subject to a given service level constraint. Cost factors are ordering cost, holding cost, purchasing cost, and shipping cost. Since we assume that no quantity discounts are available, the long run average purchasing cost will be the same under different strategies. Hence, the purchasing cost is not taken into account. The inventory on hand of item i is charged at a rate h_i per unit per unit time. If item i is included in the order, a fixed ordering cost a_i is charged.

Two options are available to ship the items from overseas to the central warehouse. The first option is to use a full-container load (FCL). In this case a fixed shipping cost F is charged, regardless of which items are included and regardless of how much of the items is shipped. The second option is a less-than-container load (LCL). Now, the shipping cost is entirely variable: c_L dollars are incurred per shipped m^3. The capacity for both FCL and LCL shipments is restricted to K m^3. Economies of scale result from the fact that $K \cdot c_L > F$. The shipping rate is a function of the volume shipped. To make this clear let O denote the vector of order quantities $(o_1, .., o_N)$, and let V(O) denote the volume of this order. ($V(O) = \Sigma_i o_i w_i$, where w_i denotes the volume of item i.) Then it is clear that a FCL is preferable if

$$\frac{F}{c_L} \leq V(O) \leq K . \tag{7.1}$$

The shipping rate per m^3, r(O), is constant (equal to c_L) until the volume exceeds F/c_L. For shipments larger than F/c_L, the shipping rate per m^3 equals $F/V(O)$. The relation between the shipping rate per m^3 and the volume shipped is shown graphically in Figure 7.1.

It is assumed that one container is enough to ship the required goods. This is not restrictive, since the extension to more containers is straightforward when each additional container required for shipment is charged according to the same price schedule.

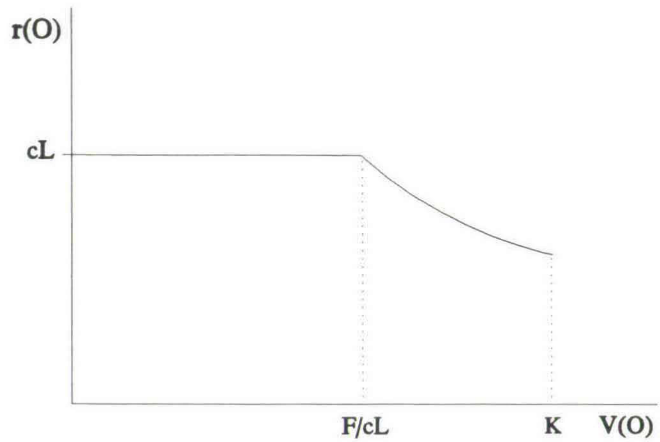

Figure 7.1 Shipping rate per m³ as a function of the volume shipped

We consider the following *hierarchical* periodic policy: management has decided that inventories are controlled by a coordinated (R,S) policy. Under this type of strategy the inventory position of a particular item i is raised up to the order-up-to level S_i every R periods. To achieve joint replenishments the review period is equal for all items. For a given value of R, the order-up-to level of item i (i=1,..,N) is the smallest value S_i for which the service level constraint is satisfied. Several authors have suggested procedures to calculate the parameter S_i (see e.g. Hadley and Whitin (1963), Naddor (1976), Silver and Peterson (1985), or De Kok (1991)). A formal algorithm is given in Appendix 7.1. Starting from the basic (R,S) inventory control policy, we propose a strategy that, in addition to holding and ordering cost, takes account of the special shipping cost structure. The decisions are based on a trade-off between holding cost, ordering cost, *and* transportation cost. In summary, we propose the following approach:

RS^+ system: (R,S) policy with joint ordering and transportation decisions

Step 1: Given the review period R, compute the optimal order-up-to level S_i for each item i (e.g. with the method of De Kok (1991)).

Step 2: At each multiple of R, the composition of the order is determined via a one-stage optimization procedure, which incorporates the potential for exploiting economies of scale in the transportation cost.

7.3 A computational procedure

At a review time, *initial order quantities* for item i are obtained by the parameters of the basic (R,S) policy: if I_i denotes the inventory position of item i, then the initial order quantity is given by $q_i := \max\{0, S_i - I_i\}$. The problem for the retailer is to decide whether the initial order quantities, denoted by the vector $Q := (q_1, .., q_N)$, have to be enlarged with $E := (e_1, .., e_N)$ units to take advantage of the lower rate per m^3 of the FCL. The decision to enlarge the initial order Q by an extra order E not only affects the ordering, shipping, and holding cost in the current review period, but also in future review periods, because the inventory positions at the following review time(s) depend on the enlargement at the current review time. Now define,

$\Delta C_1(Q,E)$: sum of the total expected *extra* ordering and holding cost when at the current review time Q+E is ordered instead of Q;

$\Delta C_2(Q,E)$: expected total *saved* shipping cost when at the current review time Q+E is ordered instead of Q.

Note that it only makes sense to enlarge the initial order Q when $V(Q+E) \geq F/c_L$ because otherwise no savings are obtained on the shipping cost (see Figure 7.1). The order will only be enlarged if the resulting order is shipped in a FCL. Now, given the actual inventory position I_i and the initial order quantity q_i for all items i, the following non-linear knapsack problem (**KS**) has to be solved to determine the optimal order composition *given that a FCL will be used.*

Problem KS

$$\min_{E} \{ \Delta C_1(Q,E) - \Delta C_2(Q,E) \}, \tag{7.2}$$

$$\sum_{i=1}^{N} (q_i + e_i) w_i \leq K, \tag{7.3}$$

$$\sum_{i=1}^{N} (q_i + e_i) w_i \geq \frac{F}{c_L}, \tag{7.4}$$

$$I_i + q_i \leq I_i + q_i + e_i \leq UB_i \qquad , i = 1,..,N. \tag{7.5}$$

The two terms of the objective function (7.2) will be specified below. Condition (7.3) indicates that the volume of the order should not exceed the capacity of the container. Condition (7.4) reflects the requirement that the total volume must exceed the threshold value F/c_L. (Recall that a FCL is only preferable if (7.4) holds.) Finally, condition (7.5) enables the decision maker to avoid large deviations from the basic inventory control policy by specifying an upper bound UB_i on the inventory position of item i just after the replenishment.

To approximate the expected extra ordering and holding cost for item i, we use the so-called *relative values* of the basic (R,S) strategy, which are denoted by $\nu_i(j)$, where j denotes the inventory position of item i just after a replenishment. The difference between the relative values $\nu_i(r)$ and $\nu_i(s)$ can be interpreted as the difference in the expected ordering and holding cost of item i over an infinitely long period when starting with inventory position r instead of s. (See also Chapter 4.) The determination of the relative values $\nu_i(j)$ for a given (R,S) strategy is discussed in Appendix 7.1. Now, $\Delta C_1(Q,E)$ is approximated by

$$\Delta C_1(Q,E) = \sum_{i=1}^{N} \psi_i(q_i,e_i) \;, \tag{7.6}$$

where,

$$\psi_i(q_i,e_i) = \begin{cases} \nu_i(S_i+e_i) - \nu_i(S_i) & \text{if } q_i>0 \\ \delta(e_i)a_i + \nu_i(I_i+e_i) - \nu_i(I_i) & \text{if } q_i=0 \;. \end{cases} \tag{7.7}$$

Note that, at a review time, the inventory position of item i can be larger than S_i because of an enlargement of the initial order quantity at an earlier review time. In such a case, the initial order quantity of item i equals zero and hence this item will not be ordered unless $e_i > 0$. The term $\delta(e_i) \cdot a_i$ is included in (7.7) because extra fixed ordering costs are incurred in case the initial order quantity is zero while $e_i > 0$.

Next, we analyze the second term in the objective function. Recall that the shipping rate per m^3 will decrease when the initial order Q is enlarged to Q+E (and condition (7.3) and (7.4) hold). It is easy to calculate the expected saved shipping cost corresponding to the initial order quantities. The problem, however, is to determine the saved expenses due to the extra ordered units. These units would have been ordered at a following review time, and the savings depend on the shipping rate at that moment. To approximate $\Delta C_2(Q,E)$, we assume that the extra units would have been shipped at a fixed rate of c_A at (one of) the following review time(s). Here, c_A denotes the average shipping rate when using the integrated inventory control and transportation planning system. (Note that $F/K \le c_A \le c_L$.) In practice, the inventory planner can use historical data to obtain an estimate for the average shipping rate in the future. The parameter c_A is updated after every order. Now, the saved shipping cost is approximated by

$$\Delta C_2(Q,E) \approx \begin{cases} c_A \sum_{i=1}^{N} e_i w_i & \text{if } \sum_{i=1}^{N} q_i w_i \ge \frac{F}{c_L}, \\ c_A \sum_{i=1}^{N} e_i w_i + c_L \sum_{i=1}^{N} q_i w_i - F & \text{if } \sum_{i=1}^{N} q_i w_i < \frac{F}{c_L}. \end{cases} \tag{7.8}$$

The knapsack problem **KS** can be solved by standard dynamic programming techniques. However, because of the possible large state space, we suggest to use a greedy heuristic such as described in Appendix 7.2. Recall that the solution of the knapsack problem provides the optimal order composition when a FCL is used. Of course, also the possibility to order the initial order quantities only has to be taken into account. Our solution procedure is summarized in the following algorithm.

Algorithm for (R,S) policy with joint ordering and transportation decisions

Step 1: Given the review period R, compute the optimal order-up-to level S_i for each item i (e.g. with the method of De Kok (1991)).

Step 2: At each review time:

a) Determine for each item its present inventory level I_i.
b) Determine the initial order sizes: $q_i := S_i - I_i$ if $I_i < S_i$; $q_i := 0$ otherwise. If $q_i = 0$ for all items i, then order nothing; else go to Step 2c.
c) If $\Sigma_i (UB_i - I_i)\, w_i < F/c_L$, then order q_i, $i=1,..,N$ (threshold, above which a FCL is preferable, cannot be reached); else go to step 2d.
d) Solve **KS** yielding a vector e_i^*, $i=1,..,N$, and an objective function value, defined by $C(e_1^*,..,e_N^*)$.
e) If $C(e_1^*,..,e_N^*) < 0$, then order $q_i + e_i^*$ of item i (enlarge the initial order sizes); else order q_i of item i $(i=1,..,N)$.

The review period R is quite often determined by external factors. However, in other cases, the inventory planner is free to choose among a limited number of possible review periods (e.g. one, two, or three weeks). Note that the review period on one hand affects the ordering and holding cost, corresponding to the basic (R,S) policy, and on the other hand affects the transportation cost by the initial order quantities. If the relation between inventory control and transportation planning is ignored, the inventory planner will choose the review period corresponding to the optimal (R,S) policy. So, the review period is based on a trade-off among ordering cost, holding cost, and the required service level. However, if the dependence between these logistics functions is taken into account explicitly, then the inventory planner will use the review period for which the sum of the holding, ordering, and transportation cost is minimal. Simulation can be used to perform an optimization with respect to R for the integrated system which uses the approach of this section.

Once the review period has been set, the parameters of the (R,S) policy, obtained in Step 1, remain constant. Step 2, however, is a dynamic procedure which uses the actual inventory levels as input. (Step 2 is carried out on line at every review time.)

The review period R is a common basic period for all items in the family. Instead of one common review period it is also possible to consider item-dependent review periods R_i. To achieve coordination, the periods R_i are then chosen as some multiple k_i of a basic period (e.g. a week). However, we consider only the case where $k_i = 1$ for all i. It is easy to adapt the method to the more general case where the review periods are not equal for all items.

An additional effect of enlarging the order quantities is an improvement of the service. However, these effects are not taken into account explicitly in the optimization.

Note that the hierarchical approach is quite similar to the one which has been proposed in Chapter 4 for the coordinated replenishment problem with all-units quantity discounts and joint ordering costs. The approaches differ in the basic ordering strategy which is used. (A (R,S) policy is used in this chapter, whereas a can-order policy is used in Chapter 4.) It is shown in Appendix 7.3 that the procedure which is described in this section can easily be adapted to handle also the joint replenishment problem as considered in Chapter 4.

7.4 Numerical examples

In this section we show how our strategy performs on a set of test problems. Simulation will be used to obtain the long run average expected total cost per period. The cost of two different strategies will be compared:

- *strategy S1*: (R,S) policy with no joint ordering and transportation planning;
- *strategy S2*: (R,S) policy with joint ordering and transportation planning.

Under strategy S1 a LCL is used whenever $V(Q) < F/c_L$ and a FCL otherwise. Changing the initial order quantities is not allowed. Strategy S2 uses the method of Section 7.3 to decide whether the initial order should be enlarged or not at a given review time.

Table 7.1 lists the parameters for a family of five items. Demands per period for item i follow a mixed-Erlang distribution with mean μ_i and variance σ_i^2. The service level requires that at least 95% of demand is satisfied directly from inventory on hand. Further, management has decided that the inventory position of item i should not exceed the expected demand for seven periods; hence $UB_i = 7 \cdot \mu_i$. The capacity of the container equals 500 m^3.

With respect to the review time and the lead time, we consider two cases:

case (a): R=2, L=1, *case (b)*: R=1, L=2.

The order-up-to level of item i, for a given value of R and L, is the smallest value S_i for which the service level is satisfied. Under strategy S1 the expected average volume that has to be transported is equal to 270 and 135 for case (a) and (b), respectively.

Table 7.1 Data for numerical examples

i	μ_i	σ_i	w_i	h_i	a_i	case (a)	case (b)
						S_i	S_i
1	5	4	1.0	1.0	10	25	27
2	10	8	2.0	2.0	20	49	54
3	15	5	3.0	3.0	30	51	54
4	10	4	4.0	2.0	40	35	38
5	5	2	5.0	1.0	50	18	19

In the simulation experiments the variable LCL transportation cost (c_L) and the break-even volume (F/c_L), above which a FCL is preferred, are both varied over three levels. For each combination of c_L and F/c_L simulation runs are repeated until a 95% confidence interval is obtained for the total average cost per period with a bandwidth of 5. A single run consists of simulating the multi-item system for 1000 review periods. The inventory control system is simulated simultaneously for strategy S1 and S2. So common random numbers (demands for the items) are used for the evaluation of the performance of both strategies. The results are reported in Table 7.2. The simulated average cost per period for strategy S1 and S2 are denoted by S1 and S2, respectively. The percentage cost saving of using strategy S2 instead of strategy S1, denoted by % c.s, is defined as 100 · (S1-S2)/S1.

Table 7.2 Results for numerical examples

F/c_L	c_L	case (a): R=2, L=1			case (b): R=1, L=2		
		S1	S2	% c.s	S1	S2	% c.s.
300	2	514	479	6.8	574	512	10.7
350	2	520	494	4.9	574	532	7.3
400	2	520	507	2.5	574	548	4.4
300	4	778	673	13.5	844	695	17.7
350	4	788	713	9.6	845	739	12.5
400	4	790	748	5.3	845	780	7.9
300	6	1042	854	18.0	1112	876	21.2
350	6	1057	921	12.9	1113	942	15.3
400	6	1059	982	7.2	1113	1009	9.4

The simulation results show that strategy S2 outperforms strategy S1 significantly in several test cases. It turns out that the percentage cost saving decreases as F/c_L increases while c_L is kept fixed. This could be expected because the potential cost saving from economies of scale decreases when the difference ($K\text{-}F/c_L$) decreases. Table 7.2 also shows that the percentage cost saving from using strategy S2 instead of S1 increases if c_L increases while F/c_L remains constant. This can be explained by the fact that the proportion of the transportation cost in the total cost increases in case c_L increases and therefore reductions on this cost factor have a larger impact on the total cost. In comparing case (a) and case (b) we conclude that the observations which are mentioned above hold for both $R>L$ and $R<L$.

The choice of the review period R has been discussed in Section 7.3. The review period R is quite often determined by external factors, but, in other cases, the inventory planner is free to choose among a limited number of possible review periods. Under strategy S1, the inventory planner will choose the review period corresponding to the optimal (R,S) policy. So, the review period is based on the sum of the ordering and holding cost. However, strategy S2 accounts explicitly for the relation between inventory control (the basic (R,S) policy) and transportation planning (the choice of using a FCL or LCL). Now, the inventory planner will use the review period for which the sum of the holding, ordering, *and* transportation cost is the lowest, when the approach of Section 7.3 is used.

Table 7.3 gives the results for the family of Table 7.1 when the lead time L equals one period. Possible values of the review period are one, two, or three periods. It turns out that the optimal review period equals two under strategy S1, whereas under strategy S2 the optimal review period is one, two, or three, depending on the value of F and c_L. Note that the cost differences under strategy S2 are very small for different values of the review period.

Table 7.3 Cost of strategy S1 and S2 for different review periods (L=1)

F/c_L	c_L	strategy S1	strategy S2			
		$R^*=2$	R=1	R=2	R=3	R^*
300	2	514	483	479	476	3
350	2	520	499	494	498	2
400	2	520	516	507	516	2
300	4	778	670	673	674	1
350	4	788	712	713	709	3
400	4	790	752	748	749	2
300	6	1042	849	854	846	3
350	6	1057	919	921	914	3
400	6	1059	985	982	979	3

7.5 Conclusions

In this chapter we suggested a simple method to handle the interaction between ordering and transportation decisions if economies of scale exist because of reduced freight rates when using a full-container load instead of a less-than-container load for transportation from overseas. A full-container load is achieved by coordinating the orders of different items.

The periodic review (R,S) policy is used as a basic inventory control policy for all items. In addition, a heuristic is proposed which decides whether to enlarge the initial order or not at a review time. This decision is based on a comparison of the expected saved shipping cost, the expected saved ordering cost, and the expected extra holding cost from an extra order. Numerical results show that the total cost can be substantially decreased (up to 20%) in case ordering and transportation planning are integrated.

Moreover, the service is increased by enlarging the initial order quantities.

A direction for further research originates from the observation that in practice the lead time is often shorter when using a FCL instead of a LCL. When ordering a LCL, one has to wait until the container is filled up with less-than-container loads of other retailers. Additional research is needed to handle this aspect.

Appendix 7.1 Computation of the relative values

Recall that ν_i (j) denotes the relative value of item i with an inventory position of j units, just after an order according to a fixed (R,S) policy with order-up-to level S_i for item i. (Note that, for a fixed review period R, the relative values of item i do not depend on the order-up-to levels S_j of the other items.) For convenience, the subscript i will be deleted in the notation from now on.

For a fixed (R,S) strategy, the relative values can be determined by the theory of regenerative processes. Denote by X_n the inventory position of item i just after the *n*th order. Then X_n is a regenerative process with regeneration epochs the moments at which an order is placed. (So, the regeneration state is S.) The attention can be restricted to the cost incurred between two subsequent replenishment orders for that particular item.

Now, define for any possible inventory position x (larger than or equal to S):

τ(x) : the expected time until the next replenishment order, given that at time zero the inventory position equals x;

η(x) : the expected holding cost until the next replenishment, given that at time zero the inventory position equals x;

κ(x) : the expected holding cost until the next replenishment together with the ordering cost incurred at the next replenishment, given that at time zero the inventory position equals x.

Hence,

$$\kappa(x) = a + \eta(x) . \tag{A.7.1}$$

The quantities τ(x) and η(x) can be determined by conditioning on the demand during the next review period. Denote the probability that demand during t periods equals j by ϕ_t (j). Then, for $x \geq S$,

$$\tau(x) = R + \sum_{j=0}^{x-S} \tau(x-j)\,\phi_R(j) , \tag{A.7.2}$$

and

$$\eta(x) = h\sum_{k=0}^{\infty} \phi_L(k)\,\omega_R(x-k) + \sum_{j=0}^{x-S} \phi_R(j)\,\eta(x-j) , \tag{A.7.3}$$

where,

$\omega_t(y)$: the total expected number of items on stock during t periods, given that the starting inventory equals y and no order arrives during these t periods.

The first term in (A.7.3) denotes the expected holding cost during the next review period. This cost expression is based on the well-known convention to shift the holding cost in a period [tR+L,tR+T+L] to the time interval [tR,tR+T]), where T denotes the cycle time. It is obvious that $\omega_t(y)$ equals zero if $y \leq 0$. Under the assumption that demands during one period arrive with a constant rate, $\omega_t(y)$ can be computed for $t>0$ (and $y>0$) using the following formula:

$$\omega_t(y) = \sum_{j=0}^{y} \phi_1(j)\left\{ y - \frac{j}{2} + \omega_{t-1}(y-j) \right\} + \sum_{j=y+1}^{\infty} \phi_1(j)\frac{y^2}{2j} . \tag{A.7.4}$$

Hence, $\omega_R(y)$ can be computed recursively starting with $\omega_0(y)=0$.

From the theory of regenerative processes (see Tijms (1986)), it follows that the long run average cost per period for a given (R,S) strategy, denoted by g(R,S), is given by

$$g(R,S) = \frac{\kappa(S)}{\tau(S)} , \tag{A.7.5}$$

while the relative values are given by

$$v(x) = \kappa(x) - g(R,S)\tau(x) . \tag{A.7.6}$$

Algorithm for determining $\nu(x)$

Step 1: Compute $\tau(x)$ and $\kappa(x)$ recursively from (A.7.1) up to (A.7.4) for x=S,..,UB.

Step 2: Compute g(R,S) from (A.7.5).

Step 3: Compute $\nu(x)$ for x=S,..,UB from (A.7.6).

Formula (A.7.1) up to (A.7.6) are based on a *given* (R,S) policy. For a given value of R, the optimal order-up-to level is the smallest value S^* for which the service level is satisfied. From the theory of regenerative processes, it follows that for a given (R,S) strategy, the long-run fraction of demand delivered from stock on hand, $\beta(R,S)$, is given by

$$\beta(R,S) = 1 - \frac{\alpha(S)}{Q} , \tag{A.7.7}$$

where $\alpha(S)$ denotes the expected number of shortages between two successive deliveries,

$$\alpha(S) = \frac{\sum_{j=1}^{\infty} \phi_R(j) \sum_{k=\max(0,S-j)}^{\infty} (j+k-S)\,\phi_L(k)}{1-\phi_R(0)} - \sum_{k=S}^{\infty} (k-S)\,\phi_L(k)\,, \tag{A.7.8}$$

and Q denotes the expected demand during a replenishment cycle,

$$Q = \frac{\sum_{j=0}^{\infty} j\,\phi_R(j)}{1-\phi_R(0)} = \frac{R\mu}{1-\phi_R(0)}. \tag{A.7.9}$$

Algorithm to determine S^* for a given review period R and required service level β

Step 1 : Determine Q from (A.7.9). Initialize S:=-1.

Step 2 : S:=S+1; Determine $\alpha(S)$ from (A.7.8).

Step 3 : If $1-\alpha(S)/Q \geq \beta$, set $S^*:=S$; otherwise go back to Step 2.

Denote the optimal order-up-to level and the minimal cost of item i, for a given review period R, by S_i^* and $g_i(R,S_i^*)$, respectively. The optimal review period, with respect to holding and ordering cost, is then obtained by an enumeration over the range of possible review periods: $R^* = \arg\min_R \Sigma_i\, g_i(R,S_i^*)$.

Appendix 7.2 A heuristic approach to solve the knapsack problem KS

The knapsack problem **KS**, which has been presented in Section 7.3, can be solved by standard dynamic programming techniques. However, such an approach may be rather time consuming. In our experiments we used a greedy heuristic.

First, note that objective function (7.2) of the knapsack problem is a separable function:

$$\Delta C_1(Q,E) - \Delta C_2(Q,E) = \sum_{i=1}^{N} \{\psi_i(q_i,e_i) - c_A e_i w_i\} + c\,, \tag{A.7.10}$$

where c is a constant that does not depend on e_i, i=1,..,N.

In Step 1 of the heuristic, the problem **KS** is relaxed by deleting conditions (7.3) and (7.4). Because of constraint (7.5), the order quantity e_i for item i cannot exceed $U_i := UB_i - I_i - q_i$. In Step 2, the solution is adapted, if necessary, such that the conditions

(7.3.) and (7.4) are satisfied. In Step 2 the vector E is modified based on the incremental cost per m^3 of adding or deleting one unit, denoted by Δ_i^+ and Δ_i^-, respectively. Let V denote the volume of the current order, then it readily follows that

$$\Delta_i^+ = \begin{cases} \infty & \text{if } e_i = U_i \vee V + w_i > K \,, \\ \dfrac{\psi_i(q_i, e_i + 1) - \psi_i(q_i, e_i) - c_A w_i}{w_i} & \text{otherwise} \,, \end{cases} \tag{A.7.11}$$

and

$$\Delta_i^- = \begin{cases} \infty & \text{if } e_i = 0 \vee V - w_i < \dfrac{F}{c_L} \,, \\ \dfrac{\psi_i(q_i, e_i - 1) - \psi_i(q_i, e_i) + c_A w_i}{w_i} & \text{otherwise} \,. \end{cases} \tag{A.7.12}$$

Algorithm to solve the knapsack problem KS

Step 1: Determine $e_i = \arg\min_{0 \le x \le U_i} \{ \psi_i(q_i, x) - c_A w_i x \}$, (A.7.13)

for $i=1,..,N$.

Calculate $V := \Sigma_i (q_i + e_i) w_i$.

- If $V > K$ (constraint (7.3) is violated), go to Step 2a.
- If $V < F/c_L$ (constraint (7.4) is violated), go to Step 2b.
- If $F/c_L \le V \le K$ (feasible solution), go to Step 3.

Step 2: Find a feasible solution:

a) Determine Δ_i^- for $i=1,..,N$ from (A.7.12).

Repeat until $V \le K$:

$j := \arg\min_i \Delta_i^-$; $e_j := e_j - 1$; $V := V - v_j$; calculate Δ_j^-.

b) Determine Δ^+ for $i=1,..,N$ from (A.7.11).

Repeat until $V \ge F/c_L$:

$j := \arg\min_i \Delta_i^+$; $e_j := e_j + 1$; $V := V + v_j$; calculate Δ_j^+.

Step 3: $e_i^* := e_i$, $i=1,..,N$.

Appendix 7.3 $(R,S)^+$ policy for discount evaluation

The procedure as described in Section 7.3 is also applicable for the joint replenishment problem considered in Chapter 4 (and 5). In Chapter 4 we investigated a multi-item problem with a joint ordering cost structure and quantity discounts. So, except for the (minor) individual ordering cost, a_i, for each item i, a (major) joint ordering cost, A, is charged for every family order. Another incentive for joint replenishment of several items is provided by the all-units discount structure, where a percentage discount, d, is awarded on the total dollar value, Q, of the order, if this dollar value exceeds a given threshold Q_d. The following hierarchical strategy is proposed for this situation.

RS⁺ system: (R,S) policy with discount opportunities

Step 1: Given the review period R, compute the optimal order-up-to level S_i for each item i (e.g. with the method of De Kok (1991)).

Step 2: At each review time according to the (R,S) policy from Step 1, the composition of the order is determined via a one-stage optimization procedure, which incorporates the potential for exploiting the quantity discount.

In this appendix we denote with w_i the unit purchasing cost of item i (before discounts), and let K denote the maximum purchasing value, which is set by the supplier ($Q_d \leq K \leq \infty$). Then, the following knapsack problem has to be solved to determine the optimal order quantities *given that the discount is taken.*

Problem KS

$$\min_{E} \left\{ \Delta C_1(Q,E) - \Delta C_2(Q,E) - \Delta C_3(Q,E) \right\} , \qquad \text{(A.7.14)}$$

$$\sum_{i=1}^{N} (q_i + e_i) w_i \leq K , \qquad \text{(A.7.15)}$$

$$\sum_{i=1}^{N} (q_i + e_i) w_i \geq Q_d , \qquad \text{(A.7.16)}$$

$$I_i + q_i \leq I_i + q_i + e_i \leq UB_i \quad , i = 1,..,N, \qquad \text{(A.7.17)}$$

where,

ΔC_1 (Q,E) : sum of the total expected *extra individual* ordering and holding cost when at the current review time Q+E is ordered instead of Q;

$\Delta C_2(Q,E)$: expected total *saved* purchasing cost when at the current review time Q+E is ordered instead of Q;

$\Delta C_3(Q,E)$: expected total *saved* joint ordering cost when at the current review time Q+E is ordered instead of Q.

The methodology of Section 7.3 can be used to approximate the cost factors $\Delta C_1(Q,E)$ and $\Delta C_2(Q,E)$. The first term in objective function (A.7.14) can be calculated from formula (7.6) and (7.7). The algorithm in Appendix 7.1 can still be used to determine the relative values, which are required in (7.7).

An expression for the expected saved purchasing cost is derived along the same lines as formula (7.8) for the expected saved shipping cost. We assume that a percentage discount c_B would have been obtained when the extra units had been ordered at a following review period. c_B is the average discount factor, obtained from historical data, which is realized by the RS$^+$ system ($0 \leq c_B \leq d$). So, the expected saved purchasing cost is approximated by

$$\Delta C_2(Q,E) \approx \begin{cases} (d-c_B)\sum_{i=1}^{N} e_i w_i & \text{if } \sum_{i=1}^{N} q_i w_i \geq Q_d \,, \\ (d-c_B)\sum_{i=1}^{N} e_i w_i + d\sum_{i=1}^{N} q_i w_i & \text{if } \sum_{i=1}^{N} q_i w_i < Q_d \,. \end{cases} \tag{A.7.18}$$

The third term of the objective function, $\Delta C_3(Q,E)$, accounts for the saving which occurs when a family order, and hence the joint ordering cost A, is saved at the next review time, due to extra orders at the current review time. Denote by P(O) the probability that no family order will be triggered at the next review time when the order quantities at the current review time equal $O:=(o_1,..,o_N)$, and denote the demand during the next review period by $D_{i,R}$. Then,

$$P(O) = \prod_{i=1}^{N} P\{D_{i,R} \leq (I_i + o_i - S_i)\} \,, \tag{A.7.19}$$

To calculate $\Delta C_3(Q,E)$, we use the following approximation:

$$\Delta C_3(Q,E) = A\{P(Q) - P(Q+E)\} \,. \tag{A.7.20}$$

The heuristic in Appendix 7.2 to solve the knapsack problem is based on the separability of the objective function (7.2). However, the objective function (A.7.14) is not separable because of the term ΔC_3 (Q,E). A solution for this problem could be to use Step 1 and 2 of the algorithm in Appendix 7.2 in a first stage, and then to use an incremental improvement approach in a second stage. It is easy to incorporate an extra term in Δ_i^+ and Δ_i^- to account for ΔC_3(Q,E). However, since the effect of adding or deleting one extra unit on ΔC_3 (Q,E) is small, we recommend to ignore this factor in the computation of the optimal vector E. The factor ΔC_3 (Q,E) is only used in the calculation of the objective function value $C(e_1^*,..,e_N^*)$.

Algorithm to solve the knapsack problem

Step 1: Determine $e_i = \arg\min_{0 \le x \le U_i} \{\psi_i(q_i,x) - (d - c_B)\, w_i x\}$, (A.7.21)

for i=1,..,N.

Calculate $V := \Sigma_i\ (q_i+e_i)w_i$ (V denotes the dollar value of the current order).

- If $V > K$ (constraint (A.7.15) is violated), go to Step 2a.
- If $V < F/c_L$ (constraint (A.7.16) is violated), go to Step 2b.
- If $F/c_L \le V \le K$ (feasible solution), go to Step 3.

Step 2: Find a feasible solution:

a) Determine for i=1,..,N:

$$\Delta_i^- = \begin{cases} \infty & \text{if } e_i = 0 \lor V - w_i < Q_d\,, \\ \dfrac{\psi_i(q_i,e_i-1) - \psi_i(q_i,e_i) + (d - c_B)\,w_i}{w_i} & \text{otherwise}\,. \end{cases}$$

(A.7.22)

Repeat until $V \le K$:

$j := \arg\min_i \Delta_i^-$; $e_j := e_j - 1$; $V := V - v_j$; calculate Δ_j^-.

b) Determine for i=1,..,N:

$$\Delta_i^+ = \begin{cases} \infty & \text{if } e_i = U_i \lor V + w_i > K\,, \\ \dfrac{\psi_i(q_i,e_i+1) - \psi_i(q_i,e_i) - (d - c_B)\,w_i}{w_i} & \text{otherwise}\,. \end{cases}$$

(A.7.23)

Repeat until $V \ge F/c_L$:

$j := \arg\min_i \Delta_i^+$; $e_j := e_j + 1$; $V := V + v_j$; calculate Δ_j^+.

Step 3: $e_i^* := e_i$, i=1,..,N.

For the sake of completeness, the solution procedure is summarized below.

Algorithm for $(R,S)^+$ strategy with discount opportunities

Step 1: Given the review period R, compute the optimal order-up-to level S_i for each item i (e.g. with the method of De Kok (1991)).

Step 2: At each review time:

a) Determine for each item its present inventory level I_i.

b) Determine the initial order sizes: $q_i := S_i - I_i$ if $I_i < S_i$; $q_i := 0$ otherwise. If $q_i = 0$ for all items i, then order nothing; else go to Step 2c.

c) If $\Sigma_i (UB_i - I_i) w_i < Q_d$, then order q_i, $i=1,..,N$ (threshold Q_d cannot be reached); else go to step 2d.

d) Solve **KS** yielding a vector e_i^*, $i=1,..,N$, and an objective function value $C(e_1^*,..,e_N^*)$.

e) If $C(e_1^*,..,e_N^*)<0$, then order $q_i + e_i^*$ of item i (enlarge the initial order sizes); else order q_i of item i $(i=1,..,N)$.

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CHAPTER 8

THE VALUE OF INFORMATION ABOUT THE SUPPLIER PRODUCTION RUN STATUS IN INVENTORY MANAGEMENT

This chapter deals with the situation where a supplier, who produces on order in fixed production cycles, provides information about the status of the coming production run. The retailer can use this information to gain insight into the lead-time process. A fixed order (s_t, Q) strategy is presented, with a set of reorder points s_t, depending on the time t until the first possible delivery, which is determined by the information of the supplier. A Markov model that analyzes a given (s_t, Q) strategy is used to quantify the value of the information provided by the supplier. Some numerical examples show that the approach may lead to considerable cost savings compared with the traditional approach that uses only one single reorder point, based on a two-moments approximation. The results of this research can be used to balance the pros and cons of a more frequent exchange of information between retailers and suppliers.

8.1 Introduction

Under pressure of Just-in-Time management, there is a trend towards closer relations between retailers (or manufacturers) and their suppliers. Co-makership is an extreme type of such a relation. Philips was one of the first companies in the Netherlands that established such a long term relation with a restricted number of suppliers. These co-makership relations enable Philips to concentrate only on their "core business" activities (with a high added value). In recent years, several companies have reduced the number of suppliers. (For example, Rank Xerox reduced the number of suppliers from thousands to a few hundreds.) With the choice of "favoured" or "prime" suppliers, there can be greater provision for quality, price, and inventory control. Another trend is to inform suppliers of expected annual demand. If suppliers are aware of annual needs, they can plan their

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production to have sufficient inventory available to meet expected demand. These contacts can reduce lead time and permit the supplier to better plan and schedule production operations. For example, the Dutch aircraft producer Fokker set up so-called "prognosis contracts" with some suppliers, by which estimates are given for the order moments together with a prognosis of the order quantities for the coming 1.5 years. The contract also specifies the maximal allowed deviation of a new prognosis compared to the latest one.

Based on these developments there is an increasing awareness that exchange of information in the logistics process can be beneficial to all parties involved. In fact, the justification of the introduction and development of Electronic Data Interchange (EDI) systems is based on this growing awareness. However, in general, quantification of the benefits of information interchange is not easy.

In this chapter we consider a specific form of information interchange between a supplier and a retailer. This transfer of information from the supplier to the retailer is intended to improve the retailer's knowledge about the lead time of an order. The exchange of information can be either organized on an ad-hoc base (by using telephone or fax) or it can be incorporated in a more extensive information system, set up between the supplier and the retailer. The numerical results show that considerable reduction in inventory holding costs can be obtained by the retailer. In particular public warehouses are preeminently in a position to materialize these cost effects, because coordination of inventory control of the retailer and production management of the supplier is part of their job. Actually the direct motivation for this research was provided by a Dutch public warehouser who offers clients a complete logistics package including transport and inventory management. For some specific products, the public warehouser is in contact with the supplier in Norway, who provides information regarding the status of upcoming production runs. This research seeks to quantify the saved inventory holding cost obtained by an improved inventory management due to the supplier information.

In setting optimal reorder policies for inventory management, the determination of the distribution of demand during the lead time (or lead time plus review time) is an important issue. An excellent survey of literature on this topic is given by Bagchi et al. (1984). Lead-time demand can be estimated directly, but can also be estimated by composing two constituent factors, i.e. demand per unit time and lead time, or even by composing three factors: order intensity, order size, and lead time. Many inventory models emphasize only the variability of demand and neglect the variability of lead time. In other models, uncertainty in the lead time is incorporated by fitting a theoretical distribution on the first two moments of the lead time (see e.g. Bagchi (1987), Carlson (1982), and Kottas and Lau (1979)) or on the first two moments of the demand during the lead time (see e.g. Fortuin (1980), Silver and Peterson (1985), and Tadikamalla (1984)). However, as is pointed out in several studies (see e.g. Kottas and Lau (1980) and Strijbosch and Heuts (1992)), a poor approximation of the empirical distribution of the lead time (demand) can have substantial cost consequences. An information system that gathers improved empirical information on the lead time would therefore be a very useful management tool.

In many models, the main component causing uncertainty in the lead time is the shipping time from the supplier to the stocking point. This reflects the situation where the supplier produces on stock. This research focuses on another frequently occurring situation, where the major source of uncertainty, from the retailer's point of view, is the time that elapses until the supplier starts a production run to fulfil the retailer's order. Here, the supplier produces on order. In our real-life case, retailers are in contact with their supplier, who provides information regarding the status of upcoming production runs. It will be shown how this information may be used to improve the inventory control on the retailer side.

This chapter is organised as follows. A model description is given in Section 8.2 together with a discussion of an appropriate inventory control policy, which will be denoted by the (s_t, Q) policy. In Section 8.3 the performance of a given (s_t, Q) policy is analyzed by means of a Markov model. The model enables us to quantify the value of information about the status of the next production run provided by the supplier. This value is expressed in savings on inventory holding cost for the retailer. The determination of the value of information is presented in Section 8.4 together with some numerical results. Finally, some conclusions are drawn in Section 8.5.

8.2 Model description

We consider a supplier who produces every C (integer) periods a batch of a particular product on order. Deliveries occur only at equidistant points in time 0, C, 2C,.. . A number of retailers place, independently of each other, (relatively small) orders for this product from time to time. An order, placed between time (j-1)C and time jC, is produced in the *j*th run, if this run is not full already. Otherwise, the order is delivered at time (j+1)C. The length of the production cycle (C) and the production time (T) is known to the retailers. Moreover, the supplier informs the retailers (by fax or telephone) as soon as a production run is booked up to capacity.

The aim of this research is to investigate how a particular retailer should use the information about the supplier production run status. The quantitative model, which will be proposed, enables the retailer to determine the benefits of this type of exchange of information with the supplier. These benefits can be balanced against the extra costs due to additional requirements on the information system or due to extra payments to the supplier (note that the supplier has otherwise no direct revenues of the exchange of information). The value of the information will be expressed in the reduction of the inventory holding cost of the retailer due to the improved inventory control. The attention is focused on a given retailer; it is assumed that other retailers will not alter their ordering policies.

To analyze the problem, we introduce the concept of *virtual lead times*. The virtual lead time (t) denotes the number of periods between the present time epoch and the end of the next production run, *which is not yet booked up to capacity*. The virtual lead time thus denotes the actual lead time, when an order is placed now. The virtual lead time decreases in real time by units of one and has an upward jump of size C when a message arrives, stating that the coming production run is filled up or that production has been started (see Figure 8.1).

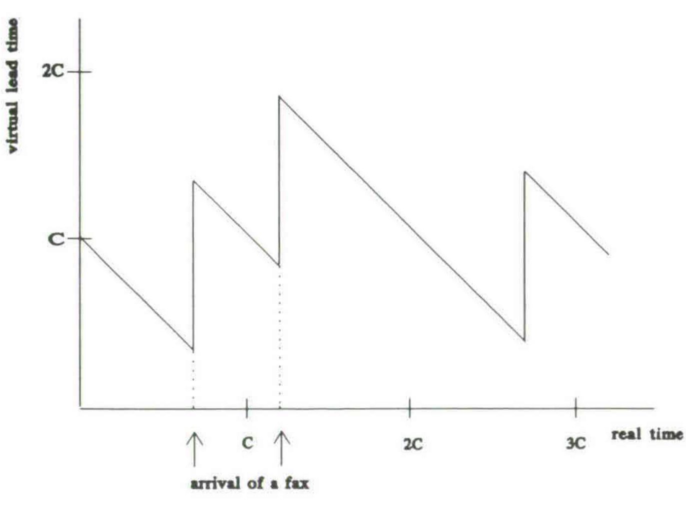

Figure 8.1 Illustration of the virtual lead time

Note that the information provided by the supplier is only partial in the sense that the retailer knows the *virtual* lead time (i.e. the actual lead time when he orders now), but not the *actual* lead time when the order is postponed for one period. There remains uncertainty about whether or not an order postponed by one additional period can be included in the next production run. Assume that the retailer and the supplier have ample experience with this information exchange system and have reached a phase of equilibrium. The retailer can then obtain statistical information from past data about the relation between the virtual lead time and the arrival of a message stating that the next production run is full.

Now define by p_t, the probability that the next production run will be filled up during the next period given a virtual lead time of t. It is assumed that $p_t=0$ for $t>C$; thus, the probability that a production run will fill up before the preceding run has been finished is negligible. The production time of an order equals an integral number of T periods ($1 \leq T \leq C$). Order entry for a production run ends as soon as the production starts. The last opportunity to place an order arises when the virtual lead time has decreased to T ($p_T = 1$). Without loss of generality, the production time is set equal to one ($T=1$) in the rest of this chapter. Shipping time from the supplier to the stocking point is neglected.

We restrict attention to fixed order quantity strategies with ordering opportunities occurring at the beginning of every period. The following ordering strategy is proposed. For every possible value t of the virtual lead time, there exists a reorder point s_t, such that a predetermined fixed quantity, Q, is ordered as soon as the inventory position drops to or below s_t. Note that in traditional models one single reorder point applies for all values of the virtual lead time. In this special situation, the dependence of the reorder points on the virtual lead times is introduced to take advantage of the information that is given by the supplier.

Demand for the product in subsequent periods is represented by independent identically distributed random variables with a discrete probability distribution, denoted by $\{ \phi_1(k), 0 \leq k \leq m_1 \}$. The mean and variance of the one-period demand are given by μ and σ^2, respectively. Excess demand is backlogged.

The retailer wants to minimize the total expected long-run average cost per time unit subject to a given service level constraint, which specifies that at least a fraction β of demand has to be satisfied directly from stock on hand. Given the structure of the ordering strategy, the retailer still must decide on the values of the reorder points s_t. Since the ordering decision can be delayed until $t=C$ without harm ($p_t = 0$ for $t>C$), only the values of s_t for t between 1 and C are important. The order quantity, Q, is not used as a decision variable, but is predetermined based on, for example, EOQ considerations or the capacity of a shipping container. The long-run purchasing and ordering cost are not affected by the choice of the decision variables. So, the only relevant cost factor is the holding cost.

The notation, introduced so far, is summarized below.

C : length of a production cycle;
T : production time (T=1);
t : virtual lead time (the number of periods between the present time epoch and the end of the next production run, *which is not yet booked up to capacity*);
p_t : the probability that the next production run will be filled up during the next period given a virtual lead time of t ($p_t=0$ for $t>C$);
$\phi_1(k)$: probability that demand during one period equals k units;
m_1 : maximal demand during one period;
μ : average demand per period;
σ^2 : variance of demand per period;
β : required fraction of demand which has to be satisfied without backorders;
Q : order quantity;
s_t : reorder point when the virtual lead time equals t.

8.3 Analysis of a given (s_t,Q) strategy

A given (s_t,Q) strategy will be analyzed as it is seen by the retailer. The performance measures of interest are the long-run average inventory level (since this determines the inventory holding cost) and the long-run fraction of demand that is satisfied directly from stock on hand (the service constraint).

Before starting with the analysis, note that in any realistic situation $s_t \geq s_{t-1} \geq 0$ for $t=2,..,C$ (reorder points will decrease with decreasing virtual lead time). It is assumed, moreover, that Q is large enough to avoid the situation that the retailer places two orders in the same production cycle.

To analyze the system, denote the inventory level of the retailer just after the *n*th arrival of a supply by X_n. Then, the embedded stochastic process $\{X_n, n=1,2,..\}$ is a discrete-time Markov chain with a state space $S:=\{ i \mid i=i_0,..,i_N \}$, where i_0 (i_N) denotes the minimal (maximal) physical inventory level under the (s_t,Q) strategy ($i_N = s_C + Q$).

For any fixed (s_t, Q) strategy, define for all states $i \in S$:

$\tau(i)$: expected number of periods until the next arrival of a supply, given that the inventory level after the present delivery equals i;

$\eta(i)$: expected total number of units on stock until the next arrival of a supply, given that the inventory level after the present delivery equals i;

$\alpha(i)$: expected number of shortages until the next arrival of a supply, given that the inventory level after the present delivery equals i.

Let p_{ij}, $i,j \in S$, denote the one-step transition probabilities of the Markov chain $\{X_n\}$ and π_i, $i \in S$, its stationary distribution. Finally, define for a given (s_t, Q) strategy:

$g(s_t, Q)$: long-run average inventory level;

$\beta(s_t, Q)$: long-run fraction of demand satisfied directly from stock on hand.

From the theory of regenerative processes (see e.g. Tijms (1986)), it follows that

$$g(s_t, Q) = \frac{\sum_{i \in S} \eta(i) \cdot \pi_i}{\sum_{i \in S} \tau(i) \cdot \pi_i}, \tag{8.1}$$

and

$$\beta(s_t, Q) = 1 - \frac{\sum_{i \in S} \alpha(i) \cdot \pi_i}{Q}. \tag{8.2}$$

Suppose that a production run has just been finished. Define:

$\phi_C(k)$: probability that demand during the next production cycle (of C periods) equals k;

$\Omega_C(i,k)$: probability that demand during the next production cycle equals k and that no order is placed during this cycle, given that the present inventory level equals i.

The determination of $\Omega_C(i,k)$, which turns out to be a key function in the derivation of explicit formulas for $\tau(i)$, $\eta(i)$, $\alpha(i)$, and p_{ij}, is discussed in Appendix 8.1.

Denote the maximal demand during C periods by m_C. Then, by conditioning on the demand during a production cycle of C periods, it readily follows that

$$\tau(i) = C + \sum_{k=0}^{m_C} \Omega_C(i,k)\, \tau(i-k) \ , \tag{8.3}$$

and

$$\alpha(i) = \begin{cases} \sum_{k=0}^{m_C} \left\{ \phi_C(k)\, [k-i,0]^+ + \Omega_C(i,k)\, \alpha(i-k) \right\} & \text{if } i>0 \ , \\ \sum_{k=0}^{m_C} \phi_C(k)\, k = \mu\, C & \text{if } i \le 0 \ , \end{cases} \tag{8.4}$$

where $[a,b]^+$ denotes the maximum of a and b. (Note that formula (8.4) is based on the assumption $s_C \geq 0$.)

It is obvious that $\eta(i)=0$ for $i \leq 0$. Using the same conditioning argument as above, it follows that

$$\eta(i) = \omega_C(i) + \sum_{k=0}^{m_C} \Omega_C(i,k)\, \eta(i-k) \qquad \text{if } \ i>0 \ , \tag{8.5}$$

where

$\omega_C(i)$: total expected number of items on stock during the next production cycle, given that the starting inventory is i.

An expression for $\omega_C(i)$ is given in Appendix 8.1.

Recall that p_{ij} denotes the probability that the inventory level of the retailer just after the next arrival of a supply equals j, given that the inventory just after the last delivery was i. These one-step transition probabilities are also derived by conditioning on the demand during a production cycle of C periods. Given that demand during one cycle equals k and that the starting inventory level is i, an order of Q units will arrive at the end of the current production cycle with probability $\phi_C(k)$-$\Omega_C(i,k)$. The inventory level just after the delivery will equal j if the demand k equals i-j+Q. Given a demand of k units, no order will be placed with probability $\Omega_C(i,k)$, in which case the inventory at the end of the production cycle will equal i-k. Hence, the following recursive relation holds:

$$p_{i,j} = \phi_C(i-j+Q) - \Omega_C(i,i-j+Q) + \sum_{k=0}^{m_C} \Omega_C(i,k)\, p_{i-k,j} \,. \tag{8.6}$$

The stationary distribution can now be obtained by the solution of the set of linear equations $\Pi=\Pi P$ (Π denotes the vector of steady state probabilities π_i and P denotes the matrix of one-step transition probabilities p_{ij}), together with the normalizing equation $\Sigma_i\ \pi_i = 1$. It is not possible to derive explicit formulas for the steady state probabilities. However, the set of equations can be solved numerically by standard procedures, such as the simple and fast iterative method of successive overrelaxation (see Tijms (1986)).

Summarizing, for a given (s_t ,Q) strategy, the quantities τ(i), α(i), and η(i) are computed recursively from formulas (8.3), (8.4), and (8.5). Once the one-step transition probabilities have been obtained by formula (8.6), the stationary distribution can be found using the method of successive overrelaxation. Finally, the long-run average inventory level and the long-run fraction of demand that is directly met from stock on hand can be computed by formulas (8.1) and (8.2).

8.4 Determination of the value of information

The Markov model, which computes the performance of a given (s_t ,Q) strategy, is now used to quantify the value of information. Recall that the value of information is expressed in savings on the inventory holding cost due to the effective use of the information about the status of the next production run.

Now suppose that the supplier provides no information about the status of the production cycles. Then, the retailer cannot do better than using a (s_t ,Q) strategy with s_t =s for t=1,..,C. (Note that the retailer has no notion about what t or p_t is.) Denoting the number of periods before the end of the *current* production cycle by w, w=1,..,C, the actual lead time of an order that is triggered now equals w if this production run is not yet booked up to capacity; it is w+C otherwise.

Define P_F (w) as the probability that the current production cycle is full at the beginning of period w. Since p_t=0 for t>C, P_F (C)=0. The probabilities P_F (w), w=1,..,C-1, can be obtained from

$$P_F(w) = 1 - \prod_{t=w+1}^{C} (1-p_t) \ . \tag{8.7}$$

Denote the probability that an order is triggered at the beginning of period w by $P_O(w)$. Note that the lack of information implies that the inventory process of the retailer and the production process of the supplier are independent in the long run. Hence, the trigger moments of orders are uniformly distributed over the production cycle; i.e. $P_O(w)=1/C$ for $w=1,..,C$. Let $P_L(j)$ denote the probability of having a lead time of j periods. Then it is easily seen that

$$P_L(j) = \begin{cases} P_O(j)\,(1-P_F(j)) \ , & \text{for } j=1,..,C, \\ P_O(j-C)\,P_F(j-C) \ , & \text{for } j=C+1,..,2C-1 \ . \end{cases} \tag{8.8}$$

Further, the expected lead time, E_L, and the variance of the lead time, V_L, are given by

$$\begin{aligned} E_L &= \sum_{w=1}^{C} P_O(w) \left\{ w + C\,P_F(w) \right\} , \\ V_L &= \sum_{w=1}^{C} P_O(w) \left\{ P_F(w)\,(w+C)^2 + (1-P_F(w))\,w^2 \right\} - E_L^2 \ . \end{aligned} \tag{8.9}$$

Of course, in the situation without information, E_L and V_L cannot be obtained from formula (8.9), but they are estimated from historical records or subjective estimates by managers. Denote the expectation and variance of demand during the lead time (L) plus review time (R) by E_D and V_D, respectively, then, it is well-known (see e.g. Silver and Peterson (1985, p.297)) that

$$E_D = (R+E_L)\,\mu \quad , \quad V_D = (R+E_L)\,\sigma^2 + V_L\,\mu^2 \ . \tag{8.10}$$

(Note that $R=1$ in our analysis.)

As mentioned in the introduction, the distribution of the demand during the lead time or during the lead time plus review time is usually approximated by fitting a suitable probability density function on the first two moments of the empirical probability distribution function. The calculation of the reorder point in the inventory system is then based on this theoretical distribution. Tijms and Groenevelt (1984) developed two-moments approximations of the reorder point in periodic and continuous review (s,S) inventory systems. They found that normal approximations give very good results with respect to the required service level if $V_D / E_D^2 \leq 0.25$; otherwise, good approximations may be found by fitting a gamma distribution (or a mixture of two Erlang distributions) to the empirical distribution. It is easy to adapt the method of Tijms and Groenevelt, which also takes account of undershoots of the reorder point, for periodic (s,Q) inventory systems. This method will be used to obtain the reorder point $\hat{s}$, which will be used in case the supplier provides no information. The average inventory level under this $(\hat{s}_t, Q)$ strategy, with $\hat{s}_t = \hat{s}$ for $t=1,..,C$, can be obtained by the Markov model.

Denote the fixed order strategy that makes the most effective use of the information of the supplier by (s_t^*, Q), and denote the holding cost per unit per period by h. *The value of information*, V_I, which is expressed in savings on the holding cost, is then given by

$$V_I = h\left\{g(\hat{s}_t, Q) - g(s_t^*, Q)\right\}, \tag{8.11}$$

To get more insight into the magnitude of the value of information, some numerical examples are presented. The following situations are considered:

- the length of the production cycle $C=2$;
- the mean demand per period equals $\mu=4$ with variance σ^2; the standard deviation σ is varied over three levels ($\sigma=1, 2, 4$);
- demands per period follow a mixed-Erlang distribution if $\sigma/\mu > 0.5$ and a normal distribution otherwise;
- the production time $T=1$; so $p_1 = 1$; the value of p_2 is varied over three levels ($p_2 = 0.2, 0.5, 0.8$);
- the predetermined order quantity is varied over two levels ($Q=20, 40$);
- the service level requires that at least 95% of demand is met directly from stock on hand ($\beta=0.95$).

The Markov model from the preceding section is used to compare the long-run average inventory levels of two strategies:

- *strategy S1*: $(\hat{s}_t, Q)$ strategy, with $\hat{s}_t = \hat{s}$ for $t=1,2$, where $\hat{s}$ is based on the two-moments approximation of Tijms and Groenevelt;
- *strategy S2*: (s_t^*, Q) strategy, where s_t^*, $t=1,2$, is a vector containing the optimal reorder points; the set of optimal reorder points is obtained by an efficient search procedure, which makes use of the Markov model.

Altogether, 18 examples are examined. A detailed list of numerical results is presented in Table 8.1. The performance of strategy S2 is measured by the percentage (cost) saving on inventory on hand, which can be obtained by an effective use of the information:

$$\%c.s. = 100 \cdot \frac{g(\hat{s}_t,Q) - g(s_t^*,Q)}{g(\hat{s}_t,Q)} = 100 \cdot \frac{V_I}{h \cdot g(\hat{s}_t,Q)} . \tag{8.12}$$

Table 8.1 shows that the percentage cost saving can be very large (up to 30%). This result corresponds with other studies that pointed out that a misspecification of the distribution of the (demand during the) lead time can have severe consequences. Note the average inventory level $g(s_t^*,Q)$ does not change substantially when varying the value of p_2. This implies that the system is quite insensitive to inaccurate estimations of p_2.

Table 8.1 Numerical results

			strategy S1			strategy S2			percentage saving on holding cost
σ	Q	p_2	$\hat{s}_1$	$\hat{s}_2$	$g(\hat{s}_t,Q)$	s_1^*	s_2^*	$g(s_t^*,Q)$	
1	20	0.2	13	13	16.7	7	12	12.6	24.7
1	20	0.5	15	15	17.5	7	12	12.6	28.3
1	20	0.8	17	17	18.3	8	12	12.6	31.2
1	40	0.2	9	9	22.7	6	10	20.8	8.3
1	40	0.5	12	12	24.5	6	10	20.7	15.5
1	40	0.8	14	14	25.3	7	10	20.7	18.0
2	20	0.2	14	14	17.7	9	13	14.1	20.1
2	20	0.5	16	16	18.5	10	13	14.2	23.3
2	20	0.8	19	19	20.3	0	14	14.5	28.4
2	40	0.2	10	10	23.7	6	11	21.9	7.8
2	40	0.5	13	13	25.5	7	11	21.9	14.1
2	40	0.8	15	15	26.3	9	11	22.0	16.5
4	20	0.2	20	20	23.5	16	18	20.3	13.5
4	20	0.5	23	23	25.2	14	20	20.9	17.4
4	20	0.8	25	25	26.1	0	21	21.4	17.9
4	40	0.2	15	15	28.8	12	13	26.1	9.2
4	40	0.5	17	17	29.6	12	15	26.6	10.2
4	40	0.8	19	19	30.4	9	16	26.8	11.9

Before analyzing the influence of the model parameters, we will decompose the value of information. Two effects can be distinguished:

- E1: effect of using the information to obtain an accurate approximation of the lead-time distribution;
- E2: effect of using different reorder points for different values of the virtual lead time.

Effect E1

The messages of the supplier provide the data for the estimation of the values of p_t ($t=1,..,C$). Once these value have been estimated, the empirical lead-time distribution can be approximated accurately (see formula (8.8)). Demand during lead time (plus review time) can be approximated directly (as Tijms and Groenevelt do), or it can be decomposed into two components: demand per time unit and lead time. Using the more detailed information about the lead time, instead of only its first two moments, may decrease the reorder point $\hat{s}$, while the required service level is still satisfied.

Now, consider

- *strategy S3*: $(\bar{s}_t, Q)$ strategy, with $\bar{s}_t = \bar{s}$ for all t, where $\bar{s}$ is the smallest reorder point for which the required service level is achieved.

Note that the difference between strategy S3 and S1 is that under S3 the information about the values of p_t is explicitly used. Starting with $\bar{s} = \hat{s}$ (this is strategy S1), the Markov model can be used to evaluate the number of units $\bar{s}$ can be decreased without violating the service level constraint.

Table 8.2 lists the values of $\bar{s}_t$, $t=1,2$, and $g(\bar{s}_t, Q)$ for the 18 examples. The difference $h \cdot \{g(\hat{s}_t, Q) - g(\bar{s}_t, Q)\}$ gives the value of an accurate approximation of the probability distribution function of the lead time.

Effect E2

The information of the supplier is also used to support the operational ordering process. The contact with the supplier enables the retailer to differentiate between situations where the present production cycle has already been booked up to capacity or not. In the former case, the lead time is C periods more than in the latter case. Thus, in the daily operations, different reorder points are used for different values of the virtual lead time. The additional value of using the information of the supplier on an operational level is given by the difference $h \cdot \{g(\bar{s}_t, Q) - g(s_t^*, Q)\}$.

Table 8.2 decomposes the percentage holding cost saving of using strategy S2 instead of S1 into the effects E1 and E2. It appears that, as expected, the effect E1 has the largest impact.

Table 8.2 Decomposition of the percentage cost saving in two effects (E1 and E2)

			strategy S3			percentage saving on holding cost		
σ	Q	p_2	$\bar{s}_1$	$\bar{s}_2$	$g(\bar{s}_t, Q)$	total	E1	E2
1	20	0.2	10	10	13.7	24.7	17.9	6.8
1	20	0.5	11	11	13.5	28.3	22.7	5.6
1	20	0.8	12	12	13.3	31.2	27.3	3.9
1	40	0.2	8	8	21.7	8.3	4.4	3.9
1	40	0.5	9	9	21.5	15.5	12.1	3.4
1	40	0.8	10	10	21.3	18.0	15.7	2.3
2	20	0.2	11	11	14.7	20.1	16.8	3.3
2	20	0.5	13	13	15.5	23.3	16.1	7.2
2	20	0.8	14	14	15.3	28.4	24.5	3.9
2	40	0.2	9	9	22.7	7.8	4.2	3.6
2	40	0.5	10	10	22.6	14.1	11.6	2.5
2	40	0.8	11	11	22.4	16.5	15.1	1.4
4	20	0.2	17	17	20.5	13.5	12.6	0.9
4	20	0.5	19	19	21.3	17.4	15.7	1.7
4	20	0.8	21	21	22.1	17.9	15.2	2.7
4	40	0.2	13	13	26.8	9.2	6.9	2.3
4	40	0.5	15	15	27.6	10.2	6.7	3.5
4	40	0.8	16	16	27.4	11.9	9.7	2.2

Table 8.3 summarizes the average percentage cost saving for fixed values of the model parameters p_2, Q, and σ. It appears that the percentage cost saving of using S2 instead of S1 increase as p_2 increases, while the other factors are kept the same. This can be explained by the fact that E_L, V_L, and the standard deviation of the lead time increase as p_2 increases. With increasing σ, the variability in demand gets more important in comparison with the variability in the lead time. Hence, it is not surprising to see that the percentage cost savings decrease as the coefficient of variation of demand increases. Table 8.3 also shows that the size of the order quantity has a large impact on the percentage saving. A larger order quantity leads to fewer orders per unit time, such that the effect from better lead time information will be significantly smaller.

Table 8.3 Average percentage cost saving

p_2	total	E1	E2
0.2	14.0	10.5	3.5
0.5	18.1	14.2	3.9
0.8	20.7	17.9	2.8
σ			
1	21.0	16.7	4.3
2	18.4	14.7	3.7
4	13.4	11.1	2.3
Q			
20	22.8	18.8	4.0
40	12.3	9.6	2.7

On an average, 35 CPU seconds were needed on a VAX-8700 computer to obtain the optimal (s_t^* ,Q) strategy in our examples. This computer time will certainly increase if the value of C increases. Note, however, that the attention can be restricted to strategies of type S3 for larger values of C, since most of the benefits appear to arise from the increased knowledge about the lead-time distribution.

8.5 Conclusions

A method has been developed for computing the value of information that is provided by a supplier who produces on order in fixed production cycles. The information of the supplier regarding the status of upcoming production runs enables retailers to improve their inventory policy by specifying a set of reorder points corresponding to the virtual lead time. The special structure of the lead time process is taken into account explicitly by means of a Markov model, which determines the performance of a given ordering strategy.

Numerical results have shown that retailers can achieve large inventory holding cost savings by an exchange of information with their suppliers, which results in a better approximation of the lead time distribution. The results of this research can be used to balance the reduced inventory cost against the increased cost due to extra requirements on the information system or the higher price charged by the supplier for the information service. The exchange of information can be either organized on an ad-hoc base (by using cheap types of transfer modes, such as telephone or fax) or it can be incorporated in a more extensive supplier-retailer information system dealing with many products purchased from the same supplier. In both cases, the marginal costs for a specific product will be rather low. Moreover, the organisational implications of this supplier-retailer relation are rather small compared to other relations such as co-makership or prime-vendorship.

The competitive advantage obtained by close supplier relationships has received little attention in the inventory management literature. The approach proposed in this chapter is a first step towards the quantification of the benefits of such contacts between retailers and their suppliers. Future research should be directed to the quantification of the value of information of other supplier-retailer relationships.

Appendix 8.1 Determination of Ω_C (i,k) and ω_C (i)

To determine an expression for $\Omega_C(i,k)$, define:

$\phi_t(k)$: probability that demand during the next t periods equals k;

$\Omega_t(i,k)$: probability that demand during the next t periods equals k and no order is placed during this time interval, given that the present inventory level equals i and the virtual lead time equals t.

$\Omega_C(i,k)$, $i \in S$ and $k=0,..,m_C$, can be computed from the following recursive relation:

$$\Omega_t(i,k) = \begin{cases} 0 & \text{if } i \le s_t\,, \\ p_t\phi_t(k) + (1-p_t)\sum_{j=0}^{k}\phi_1(j)\,\Omega_{t-1}(i-j,k-j) & \text{if } i>s_t\,,\ i-k \le s_{t-1}\,, \\ \phi_t(k) & \text{if } i>s_t\,,\ i-k>s_{t-1}\,. \end{cases} \tag{A.8.1}$$

with

$$\Omega_1(i,k) = \begin{cases} 0 & \text{if } i \le s_1\,, \\ \phi_1(k) & \text{if } i > s_1\,. \end{cases} \tag{A.8.2}$$

To explain the expression for $\Omega_t(i,k)$, consider the special case of t=2:

$$\Omega_2(i,k) = \begin{cases} 0 & \text{if } i \le s_2\,, \\ p_2\phi_2(k) + (1-p_2)\sum_{j=0}^{i-s_1-1}\phi_1(j)\,\phi_1(k-j) & \text{if } i>s_2\,,\ i-k \le s_1\,, \\ \phi_2(k) & \text{if } i>s_2\,,\ i-k>s_1\,. \end{cases} \tag{A.8.3}$$

If $i \le s_2$, then an order will be placed at the beginning of the first period, and $\Omega_2(i,k)$ equals zero, regardless of the demand k. Now, focus on the situation that $i > s_2$. In case $i-k > s_1$, no order will be placed during the next two periods, and $\Omega_2(i,k)$ equals $\phi_2(k)$. If $i-k \le s_1$, then two different scenarios can be distinguished. Firstly, the production run is filled during the first period, which occurs with probability p_2. In this case, no order will be triggered during the next two periods, and $\Omega_2(i,k)$ equals $p_2\,\phi_2(k)$. Secondly, consider the case that the production run is not filled during the first period (this occurs with probability $1-p_2$). Then, the inventory level at the beginning of the second period equals $i-j$ with probability $\phi_1(j)$, $0 \le j \le k$. The retailer will not order if $i-j > s_1$. The contribution to $\Omega_2(i,k)$ is then $(1-p_2)\phi_1(j)\,\phi_1(k-j)$.

To compute $\omega_C(i)$, define

$\omega_t(i)$: total expected number of items on stock during the next t periods, given that the starting inventory is i and no order arrives during these t periods.

Under the condition that demands occur at the end of the period, $\omega_t(i)$ equals

$$\omega_t(i) = \begin{cases} 0 & \textit{if } i \le 0\,, \\ i + \sum_{j=0}^{i} \phi_1(j)\,\omega_{t-1}(i-j) & \textit{if } i > 0\,. \end{cases} \tag{A.8.4}$$

Hence, $\omega_C(i)$ can be computed recursively starting with $\omega_1(i)=i$ if $i>0$ and $\omega_1(i)=0$ otherwise.

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SUMMARY

The cost of holding and control of inventories represent a considerable amount of the total investment and operating costs in an organisation. In addition the management of inventories directly affects the customer service. Inventory management, or more in general logistics, is therefore a critical success factor, which can spell the difference between success and failure in the market.

An inventory system provides the organizational structure and the operating policies for maintaining and controlling goods to be stocked. The inventory system is primarily concerned with two fundamental questions in inventory control: (i) when should orders be placed, and (ii) how much should be ordered ? Quantitative models may support the complex decision making in this area. The complexity is due to the large number of individual items, for which the inventory control can not be considered independently. Nevertheless, the main part of inventory management literature considers independent replenishment of a single item, whereas joint replenishments are common practice in real-life procurement processes. Coordination of replenishments of a group of items makes sense when these items are purchased from the same supplier or share the same mode of transportation. A coordinated replenishment system may lead to reduced ordering costs, reduced freight rates, reduced handling costs, quantity discounts, and improvement of the implementation of stock control.

In this thesis we analyze and compare some existing models and add some new models for coordinated control. The objective of the study is to support managerial decision making in practical situations. Therefore, attention is focused on control rules which are on one hand good enough (in the sense that they are quite close to the optimal rule), and on the other hand are easy to implement. The design and analysis of these replenishment policies are based on a combination of heuristic thinking, mathematical insights, and application of techniques of Operations Research.

Chapter 1 gives an overview of different types of inventory control systems. First of all, it differentiates between dependent demand systems and independent demand systems. Dependent demand systems assume that the demand for an item is directly related to the demand for other items. Dependent demand particularly occurs among items at different levels in the goods flow in an assembly or component industry. In this thesis attention is focused on *independent demand systems* that assume that the demand for an item is independent of the demand for any other item. Applications of such inventory

systems can specifically be found in a non-production environment, such as distribution organisations, retailers, wholesalers, as well as service industries. The main aim of this kind of control systems is to reduce the inventory related costs (such as purchasing costs, ordering costs, transportation costs, and holding costs), while maintaining a high customer service.

In practice, there are many situations where, although demand for each item is independent, it is much more natural to consider interactions among the inventory control of different items. Coordinated replenishment systems account for the cost interaction when combining orders of different items. In the literature, the cost-effectiveness of joint ordering is mostly modeled by a so-called *joint ordering cost structure*, where a joint ordering cost is incurred for any order, and an individual ordering cost is incurred for each item included in the replenishment. Section 1.4 gives an extended review of existing coordinated replenishment policies which account for this joint ordering cost structure.

Part I of the thesis considers the generalisation of the classical EOQ model, which assumes independent control of items with *constant demand*, to the multi-item situation with a joint ordering cost structure. Even in this situation, which considers arguably the most simplified coordinated replenishment system, the structure of the optimal policy can not be identified. Therefore, attention has been restricted to special classes of ordering policies, which are on one hand close to the (unknown) optimal policy and on the other hand can theoretically be analyzed and easily be implemented. In fact, the existing policies appear to fall into the class of *indirect grouping strategies* or the class of *direct grouping strategies*.

Both classes of strategies employ a fixed partition of the items into groups. Each time when an item is ordered, it is ordered jointly with the other members of its group. The time between two successive replenishments of a group is constant. The model which determines the optimal direct grouping strategy neglects the possible savings due to the synchronization of group replenishment cycles. It is assumed that no joint replenishment occurs between items that are assigned to different groups. Indirect grouping strategies explicitly account for possible savings due to synchronization of group replenishment cycles by choosing the order intervals of the groups as integer multiples of some basic cycle time. So, in contrast to direct grouping strategies, joint replenishments of different groups occur at certain multiples of the basic cycle time.

The models for the class of indirect grouping and direct grouping strategies are analyzed in ***Chapter 2***. One might conjecture that indirect grouping strategies outperform direct grouping strategies for large values of the joint ordering cost, because the corresponding model explicitly accounts for possible savings due to the synchronization of group replenishment cycles. On the other hand, indirect grouping strategies are less flexible in setting replenishment cycles, since these cycles are restricted to integer multiples of the basic cycle time. The performance of both classes of strategies, which is measured as the percentage cost saving relative to an independent strategy, is compared in an extensive numerical study. It appears that the performance depends on the *ordering cost ratio* (i.e. the ratio of the joint ordering cost and the average individual ordering cost) and the number of items. Further, it is concluded that the optimal indirect grouping strategy, in general, outperforms the optimal direct grouping strategy, except for situations where the ordering cost ratio is very small (less than 0.50) or very large (more than 50).

In the literature, a solution procedure developed by Goyal is used to find the global optimum within the class of indirect grouping strategies. In ***Chapter 3***, it is shown that Goyal's algorithm does not always lead to the optimal indirect grouping strategy. A simple correction is proposed. Numerical investigations show that the correction is necessary only when the ordering cost ratio is very small (<0.2). The cost error of using Goyal's original algorithm is rather small.

Part II describes some *stochastic* coordinated replenishment models. The literature on this type of models has almost exclusively been confined to settings where economies of scale in joint replenishments are restricted to reduced joint ordering costs. However, in practice several other incentives for coordinated control exist, such as quantity discounts and freight rate discounts.

Chapter 4 and 5 deal with two continuous review multi-item inventory systems which account for both joint ordering costs and unit-price quantity discounts. The class of *can-order policies* has extensively been studied in the literature. Savings of using can-order policies are due to reduced joint ordering costs. However, these strategies, which are simple implemented in practice, do not take discount possibilities into account. ***Chapter 4*** introduces a hierarchical policy that incorporates discount evaluation in the framework of can-order policies. An existing procedure of Federgruen, Groenevelt, and Tijms is used to compute the control parameters of the optimal can-order policy in case the demands for each item are generated by independent compound Poisson processes. This optimal can-order policy, which ignores quantity discounts, is used as a basic policy. At an epoch at which the basic policy triggers a replenishment, the composition of the order is determined via a one-stage optimization procedure, which incorporates the potential for exploiting the quantity discount. This optimization problem is solved by a simple heuristic that uses the relative values of the basic can-order policy.

Chapter 5 evaluates the performance of this system, which is referred to as the CAN$^+$ system, by comparing it with a coordinated replenishment system which has been developed by Miltenburg and Silver. Both systems are primarily developed for different demand processes: a Wiener process for Miltenburg and Silver's system and a compound Poisson process for the CAN$^+$ system. Since generalisations to other demand processes are not straightforward for both control systems (except for the simple Poisson demand process), we may conclude that Miltenburg and Silver's system is preferred for fast movers, whereas in case of erratic demand the CAN$^+$ system is preferred. In order to make a numerical comparison between both systems, the system of Miltenburg and Silver is adapted for simple Poisson demand processes. The numerical results show that the performance of the CAN$^+$ system is approximately equal to that of Miltenburg and Silver's system as far as the controllable costs are concerned. Miltenburg and Silver's system outperforms the CAN$^+$ system in case of large ordering cost ratios, whereas the CAN$^+$ system yields lower cost than Miltenburg and Silver's system for small ordering cost ratios. However, in general, the cost differences are rather small.

Chapter 6 investigates the determination of the optimal can-order policy. Traditionally, the optimal control parameters are determined by an iterative procedure which relies on a decomposition approach. It is shown that this method gives inaccurate results when the ordering cost ratio is large, because the underlying assumption for the decomposition in not valid in this case. (In some cases the model overestimates the real cost by more than 20%.) Attention is focused to a subclass of can-order policies, which is close to the optimal can-order policy for large ordering cost ratios. A heuristic solution procedure is developed to calculate the optimal control parameters of this special policy. Numerical analysis points out that this heuristic works satisfactorily.

When several items share the same transportation facility, coordination of orders may lead to reduced freight rates. ***Chapter 7*** considers a multi-item inventory system with transportation economies when ordering a full-container load instead of a less-than-container load for transportation from overseas. We propose a periodic review policy which incorporates the special transportation cost schedule into the analysis of the order quantities for a family of items. Some numerical examples show that the total cost can be substantially decreased (up to 20%) in case ordering control and transportation planning are integrated.

In ***Part III*** an approach is introduced to quantify the *value of information* in inventory management. Under pressure of the Just in Time philosophy, there is a trend towards closer relations between retailers (or manufacturers) and their suppliers. Based on this development, there is an increasing awareness that exchange of information in the logistics process can be beneficial to all parties involved. However, quantification of the benefits of information interchange is usually not easy.

Chapter 8 deals with the situation where a supplier, who produces on order in fixed production cycles, provides information about the status of upcoming production runs. Such information enables the retailers to improve their inventory control. We present a fixed order quantity policy with a set of reorder points corresponding to the prospective lead times, depending on whether the next production run is filled. A Markov model that analyses such a type of control rule, is used to quantify the value of information. The numerical examples show that the approach may lead to considerable cost savings compared with the traditional approach that uses only one single reorder point, based on a two-moments approximation of the probability distribution of the demand during the lead time and review time.

SAMENVATTING (SUMMARY IN DUTCH)

De kosten van voorraden en de beheersing hiervan vertegenwoordigen een belangrijk aandeel in de totale investerings- en operationele uitgaven van een organisatie. Een goed management van deze voorraden is dan ook van zeer groot belang. Bovendien wordt de klantenservice direct beïnvloed door beslissingen op dit gebied. Voorraadmanagement, of meer in het algemeen logistiek, wordt daarom gezien als één van de cruciale succesfactoren voor de jaren negentig.

Een bestelsysteem houdt zich primair bezig met het bepalen van wanneer welke produkten in welke hoeveelheid moeten worden besteld. Kwantitatieve modellen kunnen als nuttig hulpmiddel dienen voor de vaak zeer complexe besluitvorming op dit gebied. Deze complexiteit wordt veroorzaakt doordat de voorraad van een groot aantal produkten moet worden beheerst. Hierbij kan het voorraadbeheer van een bepaald produkt vaak niet los worden gezien van het voorraadbeheer van een groep andere produkten. Opvallend is dat de meeste modellen in de voorraadliteratuur uitgaan van onafhankelijke bestellingen van individuele produkten, terwijl in de praktijk vaak bestellingen van verschillende produkten worden gecoördineerd. Gecoördineerd bestellen kan zinvol zijn als verschillende produkten worden ingekocht bij dezelfde leverancier of als verschillende produkten gebruik maken van hetzelfde transportmiddel. Het gebruik van een gecoördineerd bestelsysteem kan leiden tot gereduceerde bestel- of transportkosten, kwantumkortingen en een verbeterde voorraadbeheersing.

Dit proefschrift analyseert en vergelijkt enkele bestaande modellen voor gecoördineerde bestelsystemen en voegt enkele nieuwe modellen toe. De modellen hebben als doel om het management in praktische situaties te ondersteunen bij de keuze van de juiste bestelstrategie. De aandacht richt zich daarom op beslissingsregels die enerzijds bijna optimaal zijn, maar anderzijds ook eenvoudig te implementeren zijn. De ontwikkeling en analyse van deze bestelstrategieën is gebaseerd op een combinatie van heuristisch denken, wiskundig inzicht en toepassing van technieken uit de Operations Research.

In ***hoofdstuk 1*** wordt een overzicht gegeven van verschillende soorten bestelsystemen. Allereerst wordt een onderscheid gemaakt tussen voorraadsystemen voor produkten met afhankelijke vraag en voorraadsystemen voor produkten met onafhankelijke vraag. Bij afhankelijke vraag is de vraag naar een produkt direct gerelateerd aan de vraag naar een ander produkt. Deze afhankelijkheid doet zich voornamelijk voor tussen produkten op verschillende niveaus in de goederenstroom in een assemblage-situatie of bij componenten-produktie. De bestelsystemen die in dit proefschrift worden beschreven gaan er echter vanuit dat de vraag naar een bepaald produkt niet wordt beïnvloed door de vraag naar een ander produkt. Toepassingen van dit soort systemen zijn vooral te vinden in niet-produktie bedrijven zoals handels- en distributiebedrijven en organisaties in de dienstensector. Het doel van deze systemen is om de aan het voorraadbeheer gerelateerde kosten (zoals inkoopkosten, bestelkosten, voorraadkosten en transportkosten) zo laag mogelijk te houden terwijl tegelijkertijd aan de klanteneisen wordt voldaan.

In de praktijk bestaan verschillende situaties waarin het, ondanks het feit dat de vraag voor ieder produkt onafhankelijk is, zinvol is om interacties tussen het voorraadbeheer van verschillende produkten in beschouwing te nemen. Gecoördineerde bestelsystemen houden rekening met de kosteninteractie tussen verschillende produkten bij een gezamenlijke bestelling. Het voordeel van een gezamenlijke bestelling wordt in de literatuur meestal gemodelleerd door een *gemeenschappelijke bestelkostenstructuur* waar bij iedere bestelling gemeenschappelijke bestelkosten, die onafhankelijk zijn van de samenstelling van de bestelling, in rekening worden gebracht, met daarbovenop individuele bestelkosten voor ieder produkt dat in de bestelling wordt meegenomen. In paragraaf 1.4 wordt een uitgebreid overzicht gegeven van gecoördineerde bestelstrategieën die rekening houden met een dergelijke gemeenschappelijke bestelkostenstructuur.

Deel I van het proefschrift beschouwt de generalisatie van het standaard EOQ model, dat uitgaat van een individueel voorraadbeleid voor produkten met een *constante vraag*, voor een gemeenschappelijke bestelkostenstructuur. Zelfs voor deze zeer vereenvoudigde weergave van de werkelijkheid is de structuur van de optimale bestelstrategie onbekend. Daarom beperkt de aandacht zich tot speciale klassen van strategieën die enerzijds bijna-optimaal zijn en anderzijds theoretisch analyseerbaar en eenvoudig implementeerbaar zijn. Deze bestelstrategieën kunnen worden onderverdeeld in de klasse van *indirecte groeperingsstrategieën* en de klasse van *directe groeperingsstrategieën*. Bij beide klassen van strategieën worden de produkten verdeeld in een aantal groepen met dezelfde constante bestelcyclus. Produkten binnen dezelfde groep worden steeds gezamenlijk besteld. In het model voor de bepaling van de optimale directe groeperingsstrategie wordt geen rekening gehouden met de mogelijke besparingen die kunnen worden behaald door op bepaalde tijdstippen verschillende groepen gezamenlijk te bestellen. Het model voor indirecte groepering houdt hier wel expliciet rekening mee door de bestelcyclus van iedere groep zodanig te kiezen dat het een geheeltallig veelvoud is van een bepaalde basiscyclus.

De modellen voor beide klassen van groeperingsstrategieën worden geanalyseerd in ***hoofdstuk 2***. Indirecte groeperingsstrategieën hebben als voordeel dat er expliciet rekening wordt gehouden met het samenvallen van groepsbestellingen, terwijl directe groeperingsstrategieën flexibeler zijn bij de bepaling van de groepsbestelcycli, omdat deze niet gebonden zijn aan een geheeltallig veelvoud van de basiscyclus. De performance van beide klassen van gecoördineerde bestelstrategieën, uitgedrukt in de procentuele kostenbesparing ten opzichte van een individuele bestelstrategie, wordt vergeleken in een uitgebreid numeriek experiment. Het blijkt dat performance afhangt van de *bestelkostenratio* (dit is de ratio van de gemeenschappelijke bestelkosten en de gemiddelde individuele bestelkosten) en het aantal artikelen in de familie. Verder wordt geconcludeerd dat de optimale indirecte groeperingsstrategie in het algemeen een betere performance heeft dan de optimale directe groeperingsstrategie, behalve in de situatie dat de bestelkostenratio erg klein is (kleiner dan 0,5) of erg groot (groter dan 50).

In de literatuur wordt een algoritme van Goyal gebruikt om de optimale indirecte groeperingsstrategie te bepalen. In ***hoofdstuk 3*** wordt aangetoond dat dit algoritme niet altijd leidt tot de optimale indirecte groeperingsstrategie. Er wordt een eenvoudige aanpassing van het algoritme voorgesteld. Uit een aantal numerieke exercities blijkt dat het gebruik van het oorspronkelijke algoritme van Goyal alleen tot een niet-optimale indirecte groeperingsstrategie leidt als de bestelkostenratio erg laag is (kleiner dan 0,2). De procentuele afwijking ten opzichte van de kosten van de optimale indirecte groeperingsstrategie is echter zeer gering.

In ***deel II*** worden enkele *stochastische* gecoördineerde bestelproblemen bestudeerd. De bestaande literatuur voor dit soort modellen beperkt zich voornamelijk tot de bestudering van situaties waarbij de voordelen van gecoördineerd bestellen veroorzaakt worden door het reduceren van gemeenschappelijke bestelkosten. Echter, in de praktijk zijn kortingen op de inkoopprijs of transportkosten, die worden gegeven bij een bepaald besteld bedrag, vaak aanleiding tot het combineren van bestellingen van verschillende artikelen.

In hoofdstuk 4 en 5 worden twee gecoördineerde bestelsystemen beschouwd die zowel gemeenschappelijke bestelkosten als kortingen op de inkoopprijs in beschouwing nemen. *Can-order strategieën* zijn een bekende klasse van strategieën die rekening houden met gemeenschappelijke bestelkosten. Echter, een dergelijke bestelstrategie, die zeer eenvoudig te implementeren is, houdt geen rekening met eventuele kortingsmogelijkheden. In ***hoofdstuk 4*** wordt een hiërarchische strategie geïntroduceerd die kortingsmogelijkheden evalueerd binnen de klasse van can-order strategieën. Een bestaande methode van

Federgruen, Groenevelt en Tijms wordt gebruikt om de parameters te bepalen van de optimale can-order strategie in de situatie dat de vraag voor ieder produkt wordt gegenereerd door een samengesteld Poisson proces. Deze "optimale" can-order strategie, die kwantumkortingen buiten beschouwing laat, wordt gebruikt als een basis-strategie. Op het moment dat de basis-strategie een bestelling veroorzaakt wordt de samenstelling van de bestelling bepaald door een optimaliseringsprocedure die eventuele kortingsmogelijkheden in beschouwing neemt. Het optimaliseringsprobeem wordt opgelost door een eenvoudige heuristiek die onder andere gebruik maakt van de relatieve waarden van de basis-strategie.

In ***hoofdstuk 5*** wordt dit nieuwe bestelsysteem, dat het CAN$^+$-systeem wordt genoemd, vergeleken met het gecoördineerd bestelsysteem dat is ontwikkeld door Miltenburg en Silver. Beide systemen zijn ontwikkeld voor verschillende vraagprocessen: Miltenburg en Silver gaan uit van een Wiener proces, terwijl het CAN$^+$-systeem uitgaat van een samengesteld Poisson proces. Aangezien een generalisatie naar andere vraagprocessen voor beide systemen niet mogelijk is (behalve voor enkelvoudige Poisson processen), kan worden geconcludeerd dat het systeem van Miltenburg en Silver te prefereren is in geval van "fast moving demand", terwijl het CAN$^+$-systeem de voorkeur geniet bij "erratic demand". Om een kwantitatieve vergelijking mogelijk te maken wordt het systeem van Miltenburg en Silver aangepast voor enkelvoudige Poisson vraag. Uit een groot aantal numerieke experimenten blijkt dat de performance van beide systemen vergelijkbaar is. Het aangepaste systeem van Miltenburg en Silver levert iets lagere kosten op bij een grote bestelkostenratio, terwijl het CAN$^+$-systeem beter presteert bij een kleine bestelkostenratio. Echter, in het algemeen zijn de kostenverschillen klein.

In ***hoofdstuk 6*** wordt de bepaling van de parameters van de optimale can-order strategie bestudeerd. In de literatuur wordt hiervoor een iteratieve heuristiek gebruikt die gebaseerd is op een decompositie van het meer-artikel probleem in een aantal één-artikel problemen. Er wordt aangetoond dat deze methode zeer onnauwkeurige resultaten oplevert als de bestelkostenratio groot is, omdat de veronderstelling die ten grondslag ligt aan de decompositie niet opgaat in deze situatie (in sommige gevallen wijken de modelkosten meer dan 20% af van de werkelijke kosten). We bestuderen een speciale klasse van can-order strategieën, die dezelfde structuur heeft als de optimale can-order strategie bij een grote bestelkostenratio. Er wordt een heuristiek ontwikkeld voor de bepaling van de optimale parameters van deze speciale can-order strategie. Uit een numerieke analyse blijkt dat de heuristiek bevredigende resultaten oplevert.

Vaak kunnen kortingen op de transportkosten per eenheid worden verkregen door artikelen die met hetzelfde transportmiddel worden vervoerd gecoördineerd te bestellen. In ***hoofdstuk 7*** wordt een voorraadsysteem beschouwd waarbij schaalvoordelen in de transportkosten kunnen optreden door het gebruik van een "full-container load" (een FCL) in plaats van een "less-than-container load" (een LCL) voor het transport over zee. We introduceren een hiërarchische strategie, die rekening houdt met de interactie tussen de bestelbeslissingen en de keuze van het type vervoer (FCL of LCL). Een periodieke aanvulstrategie wordt gebruikt als basis-strategie. Op een bestelmoment wordt er met een heuristiek nagegaan of de initiële bestelhoeveelheden moeten worden verhoogd om gebruik te maken van de mogelijke schaalvoordelen bij het gebruik van een FCL. Uit een aantal numerieke voorbeelden blijkt dat de geïntegreerde benadering van bestel- en transportbeslissingen tot grote procentuele kostenvoordelen (tot 20%) kan leiden in vergelijking met een individuele benadering.

In ***deel III*** wordt een concept ontwikkeld om de *waarde van informatie* bij voorraadmanagement te kwantificeren. Organisaties worden zich steeds meer bewust van het feit dat uitwisseling van informatie met leveranciers en afnemers grote voordelen kan opleveren. Echter, de kwantificering van de voordelen van informatie-uitwisseling is vaak erg moeilijk.

In ***hoofdstuk 8*** wordt de situatie beschouwd waarbij een leverancier, die in vaste produktie-cycli produceert, zijn afnemers informeert of een komende produktie-run reeds volgeboekt is of niet. De afnemers kunnen deze informatie gebruiken bij de bepaling van hun bestelstrategie. We introduceren een bestelstrategie met variabele bestelpunten gebaseerd op de "virtuele levertijd", die wordt verkregen uit de informatie van de leverancier. Een Markov model, dat een gegeven strategie van dit type analyseert, wordt gebruikt om de waarde van informatie te bepalen. Uit de numerieke resultaten blijkt dat deze aanpak kan leiden tot grote kostenbesparingen in vergelijking met de traditionele benadering, die slechts een enkel bestelpunt gebruikt, dat gebaseerd is op een twee-momenten benadering van de kansverdeling van de vraag gedurende de levertijd en de reviewtijd.

CURRICULUM VITAE

Marc van Eijs werd op 1 januari 1966 geboren te Sittard. Na het Atheneum te hebben doorlopen op het Bouwens van der Boye College te Helden-Panningen begon hij in 1984 aan de studie Econometrie aan de Katholieke Universiteit Brabant. Van februari 1988 tot en met september 1988 was hij werkzaam als student-assistent bij de werkeenheid Statistiek van de vakgroep Econometrie. In dezelfde periode werd het afstudeerproject uitgevoerd bij het rijksinstituut voor landschapsbouw en bosbouw "De Dorschkamp" te Wageningen. Op 23 september 1988 studeerde hij af in de variant Bedrijfseconometrische Toepassingen.

In de periode van november 1988 tot mei 1993 was hij als Assistent in Opleiding verbonden aan de werkeenheid Bedrijfseconometrie/Besliskunde. Onder begeleiding van prof. dr. F.A. van der Duyn Schouten en dr. R.M.J. Heuts is het onderzoek verricht dat heeft geleid tot dit proefschrift. Daarnaast was hij docent van een cursus voorraadmanagement en was hij, als begeleider van studenten bij afstudeerstages, betrokken bij het oplossen van diverse logistieke problemen in uiteenlopende bedrijven en organisaties.

Op 1 september 1993 treedt hij in dienst van het Academisch Ziekenhuis van de Vrije Universiteit te Amsterdam, waar hij als organisatie-adviseur werkzaam zal zijn bij het bureau managementondersteuning.